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HYPOTOPOLOGICAL SPACES AND THEIR EMBEDDINGS IN LATTICES WITH BIRKHOFF INTERVAL TOPOLOGY

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Important classes of topological spaces have topologies which are induced by a generating collection of closed subsets; typical examples are k-spaces, sequential spaces with unique sequential limits, and lattices with the Birkhoff interval topology. This paper proceeds by axiomatizing this construction —a set with a specified generating collection of closed subsets is called a "hypotopological space." The Birkhoff interval topology is then studied in these terms. A natural embedding of hypotopological spaces in conditionally complete, atomic, distributive lattices with Birkhoff interval topology is derived. This embedding is used to show that lattices with Birkhoff interval topology have the same nontrivial subspace and product properties as k-spaces and sequential spaces. In particular, we answer in the negative a question first raised by Birkhoff, namely, whether the Birkhoff interval topology is preserved under the formation of the product of two lattices.

The Birkhoff interval topology was defined in [5]. It is one of the most natural of the various topologies which have been proposed for lattices, and yet is one of the least amenable to explicit calculation. Certain cases have proven tractable: For partially ordered sets with universal bounds (0 and 1), the Birkhoff interval topology coincides with the Frink interval topology [15], under which the closed intervals form a subbase for the closed sets. In general, however, it is difficult to determine the closure of a given subset of a lattice with respect to this topology. It is for this reason that several basic questions regarding subspaces and products have proven elusive for the Birkhoff interval topology.

Subspace and product properties of k-spaces and sequential spaces have been studied by Franklin [12, 13], Cohen [9], Dowker [10], Dudley [11], Michael [19, 20], and others. Birkhoff noted that a conditionally complete lattice is itself a k-space under his interval topology [5, Th. 3]. For the Birkhoff interval topology, products of chains were studied by Alo and Frink [1]. In particular, they answered Birkhoff's question in the infinite case by showing that the Birkhoff interval topology is not preserved under infinite products of chains. The example derived in the present paper answers Birkhoff's question for the finite case (see Corollary 6.8 below).