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A CHARACTERIZATION OF THE LINEAR SETS SATISFYING HERZ'S CRITERION

HASKELL P. ROSENTHAL

Let E be a closed subset of T, the circle group, which we identify with the real numbers modulo 1. E is said to satisfy Herz's criterion (briefly, E satisfies (H)), if there exists an infinite set of positive integers N, such that

$(*)$ for all integers j with $0 \leqq j < N$, each of the numbers j/N either belongs to E or is distant by at least $1/N$ from E.

The main theorem proved here, is that E satisfies (H) if and only if there exists a sequence of sets $F_1, F_2, \cdots$ with $E = \bigcap_{i=1}^{\infty} F_i$ and positive integers $N_1 < N_2 < \cdots$ satisfying the following properties for all i:

(1) N_i divides N_{i+1} and $F_i \supset F_{i+1}$.

(2) F_i is a finite union of disjoint closed intervals each of whose end points is of the form j/N_i for some integer j.

(3) If for some integer j, $j/N_i \in F_i$, then $j/N_i \in F_{i+1}$.

The motivation for studying sets E satisfying (H) is the result of Herz (c.f. [1]) that all such sets satisfy spectral synthesis, and of course that the Cantor set is an example. (See also [2], Chapter IX).

Now suppose that $E = \bigcap_{i=1}^{\infty} F_i$, with F_i and N_i satisfying (1)–(3) for all i. It is then evident that E satisfies (H), since the numbers N_i will satisfy $(*)$ for all i. Moreover, E is obtained by a sort of disection procedure. Indeed, F_{i+1} may be obtained from F_i by removing from certain of the closed intervals $[j/N_i, (j+1)/N_i]$ included in F_i, one or more open intervals of the form

$$\left(\frac{l}{N_{i+1}}, \frac{q}{N_{i+1}}\right)$$

where $j/N_i \leqq l/N_{i+1} < q/N_{i+1} \leqq (j+1)/N_i$.

The "only if" part of our main result is demonstrated following the proof of Theorem 4 below. The latter result is somewhat stronger than our main theorem, and enables us to show that certain sets fail to satisfy (H) (in particular, the symmetric sets of ratio ξ, where ξ is a rational number with $1/\xi$ unequal to an integer. (C.f. [2], pp. 13–15 for the definition of these sets).

§ 1. *Preliminaries.* We identify the points of $\boldsymbol{T}$ with $[0, 1)$, where addition and subtraction are taken modulo 1. If x and y belong to $\boldsymbol{T}$, then the distance between them, $\rho(x, y)$, is defined to be the distance from $x-y$ to the closest integer on the real line. If E