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A RADON-NIKODYM THEOREM FOR VECTOR AND OPERATOR VALUED MEASURES

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The main result of this paper is a Radon-Nikodým theorem for measures taking values in a separable Hilbert space and on the bounded operators of such a space. The integral used for the representation is a Gelfand-Pettis integral, which in this case is also equivalent to the Bochner integral.

1.1. Basic definitions. We will consider the following objects: a measure space $(\Omega, \mathscr{A}, \mu)$, where $\mathscr{A}$ is a σ-algebra of subsets of Ω and μ is a σ-finite nonnegative measure; a separable Hilbert space H and the space $B(H)$ of bounded linear operators from H into H, and also the objects which we define below.

1.2. DEFINITION. By *vector function* and *operator function* we will understand functions defined on Ω and taking values in H and $B(H)$ respectively. A vector function $x(\omega)$ is *measurable* if for each y in H, the function $(y, x(\omega))$ is measurable. An operator function $A(\omega)$ is *measurable* if for each x, y in H, the function $(A(\omega)x, y)$ is measurable. Obviously $A(\omega)$ is measurable if and only if $A(\omega)x$ is a measurable vector function for each x in H.

1.3. LEMMA. *If $x(\omega)$ is a measurable vector function, then $\| x(\omega) \|$ is measurable. If $A(\omega)$ is a measurable operator function, then $\| A(\omega) \|$ is measurable.*

Proof. Let $x(\omega)$ be measurable and let $\{e_1 e_2, \cdots\}$ denote an orthonormal basis for H. Then $(x(\omega), e_n)$ is measurable for each n and so $\| x(\omega) \|^2 \sum_{n=1}^{\infty} |(x(\omega), e_n)|^2$ is measurable. Now let $A(\omega)$ be measurable and let S_0 be a countable dense subset of the unit ball in H. Then $\| A(\omega) \| = \sup \{\| A(\omega)x \|: x \in S_0\}$ is measurable.

1.4. DEFINITION. A measurable vector function $x(\omega)$ is *integrable* if $\| x(\omega) \|$ is integrable (i.e., it belongs to $L_1(\mu)$). A measurable operator function $A(\omega)$ is *integrable* if $\| A(\omega) \|$ is integrable.

Let $x(\omega)$ be integrable and let $y \in H$. Then $|(y, x(\omega))| \leqq \| y \| \cdot \| x(\omega) \|$ and $(y, x(\omega))$ is integrable. $\int (y, x(\omega)) d\mu(\omega)$ is a linear functional bounded by $\int \| x(\omega) \| \, d\mu(\omega)$ and there is a unique vector $z \in H$ such that $\int (y, x(\omega)) d\mu(\omega) = (y, z)$. The vector z is by definition the integral