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GEOMETRIES ON SURFACES

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Among the topological geometries, two classes have so far attracted special attention, namely the locally compact, 2-dimensional projective and affine planes. Such a plane has a pointset which is homeomorphic to the pointset of the real arguesian plane, hence is a 2-dimensional manifold. In this paper, all the 2-manifolds that can carry topological geometries with connected lines will be determined.

THEOREM. *Let M be a surface carrying a topological geometry such that any two distinct points are on a unique line. Then either*

(1) *M is an open disk, and each line is open, i.e., homeomorphic to the real line R, or*

(2) *M is a compact surface of characteristic* 1, *and each line is closed, i.e., homeomorphic to a circle, or*

(3) *M is a Moebius strip, and through each point there pass closed lines and at least one open line.*

The proof follows immediately by combining statements 1.10 and 2.3, 6, 9, 10, 13 below. From 2.9 and 2.11 we get

COROLLARY. *M is orientable if and only if the space $\mathfrak{L}$ of lines is a Moebius strip, and M is nonorientable if and only if $\mathfrak{L}$ is compact.*

1. Let $\mathfrak{L}$ be a family of subsets of a nontrivial topological space M, and assume that each two distinct "points" $p, q \in M$ are joined by a unique "line" $L = p \cup q \in \mathfrak{L}$. The system $\boldsymbol{E} = (M, \mathfrak{L})$ is called [3] a "plane" whenever $\mathfrak{L}$ can be provided with a topology (necessarily unique) for which $\boldsymbol{E}$ becomes a topological geometry in the following sense:

(a) $p \cup q$ depends continuously on (p, q), and

(b) the set $\mathfrak{D}$ of pairs of intersecting lines is open in $\mathfrak{L} \times \mathfrak{L}$, and intersection is a continuous map from $\mathfrak{D}$ onto M.

Condition (b) is equivalent to

(b′) If $U \subseteqq M$ is open, then $\{(K, L); K \cap L \in U\}$ is open in $\mathfrak{L} \times \mathfrak{L}$.

We shall be concerned with "flat" planes only, i.e., with those planes in which the underlying space M is a 2-dimensional manifold or "surface". The line space $\mathfrak{L}$ of a flat plane is also a surface, and the incident point-line pairs or "flags" form a 3-dimensional closed submanifold F of $M \times \mathfrak{L}$; each line is closed in M and is locally homeomorphic to the real line [3, 2.3].

If B is a connected open subset of M, then the system