PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 3, 1969

MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIAL AND A PRESCRIBED SUBMATRIX

GRACIANO N. DE OLIVEIRA

Let A be an arbitrary (complex) $n \times n$ matrix and let $f(\lambda)$ be a polynomial (with complex coefficients) of degree $n+1$ with leading coefficient $(-1)^{n+1}$. In this paper we solve the problem: under what conditions does there exist an $(n+1) \times (n+1)$ (complex) matrix B of which A is the submatrix standing in the top left-hand corner and such that $f(\lambda)$ is its characteristic polynomial?

In [1] Farahat and Ledermann proved that if A is a nonderogatory matrix over a field Φ and $f(\lambda)$ is a monic polynomial over Φ, then there exists an $(n+1) \times (n+1)$ matrix B over Φ with A standing in its top left-hand corner and such that $f(\lambda) = \det(\lambda E_{n+1} - B)$. Now, our main results are:

THEOREM 1. *Let A be an $n \times n$ complex matrix whose distinct characteristic roots are w_α $(\alpha = 1, \cdots, t)$. Let us suppose that in the Jordan normal form of A, w_α appears in r_α distinct diagonal blocks of orders $v_1^{(\alpha)}, \cdots, v_{r_\alpha}^{(\alpha)}$ respectively. We assume that*

$$v_1^{(\alpha)} \leqq \cdots \leqq v_{r_\alpha}^{(\alpha)} .$$

Let $\theta_\alpha = \sum_{j=1}^{r_\alpha - 1} v_j^{(\alpha)}$. There exists an $(n+1) \times (n+1)$ complex matrix B having A in the top left-hand corner and with $f(\lambda)$ as characteristic polynomial (i.e., $f(\lambda) = \det(B - \lambda E_{n+1})$) *if and only if $f(\lambda)$ is divisible by $\prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{\theta_\alpha}$.*

THEOREM 2. *Let A be a real $n \times n$ symmetric matrix whose distinct characteristic roots are w_α $(\alpha = 1, \cdots, t)$. Let r_α be the multiplicity of w_α. There exists a real $(n+1) \times (n+1)$ symmetric matrix B having A in the top left-hand corner and with $f(\lambda)$ (now with real coefficients) as characteristic polynomial if and only if*

(a) $\qquad f(\lambda)$ *is divisible by* $\prod_{\alpha=1}^{t} (w_\alpha - \lambda)^{r_\alpha - 1}$

and

(b) $$\left[\frac{f(\lambda)}{(w_\beta - \lambda)^{r_\beta - 1}}\right]_{\lambda=\lambda_\beta} \cdot \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^{t} (w_\alpha - w_\beta)^{r_\alpha} \; (\beta = 1, \cdots, t)$$

is real and nonpositive.

REMARK. There is no difficulty in seeing that the conditions (a)