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ON AN EMBEDDING PROPERTY OF GENERALIZED CARTER SUBGROUPS

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If $\mathscr{E}$ and $\mathscr{F}$ are saturated formations, $\mathscr{E}$ is strongly contained in $\mathscr{F}$ (written $\mathscr{E} \ll \mathscr{F}$) if:

(1.1) For any solvable group G with $\mathscr{E}$-subgroup E, and $\mathscr{F}$-subgroup F, some conjugate of E is contained in F.

This paper is concerned with the problem:

(1.2) Given $\mathscr{E}$, what saturated formations $\mathscr{F}$ satisfy $\mathscr{E} \ll \mathscr{F}$?

The object of this paper is to prove two theorems. The first, Theorem 5.3, shows that if $\mathscr{T}$ is a nonempty formation, and $\mathscr{E} = \{G \mid G/F(G) \in \mathscr{T}\}$. ($F(G)$ is the Fitting subgroup of G), then any formation $\mathscr{F}$ which strongly contains $\mathscr{E}$ has essentially the same structure as $\mathscr{E}$ in that there is a nonempty formation $\mathscr{U}$ such that $\mathscr{F} = \{G \mid G/F(G) \in \mathscr{U}\}$. The second, Theorem 5.8, exhibits a large class of formations which are maximal in the partial ordering $\ll$. In particular, if $\mathscr{N}^i$ denotes the formation of groups which have nilpotent length at most i, then $\mathscr{N}^i$ is maximal in $\ll$. Since for $\mathscr{N} = \mathscr{N}^1$, the $\mathscr{N}$-subgroups of a solvable group G are the Carter subgroups, question (1.2) is settled for the Carter subgroups.

Since the theory of formations is of relatively recent origin, we give a few highlights. The theory begins with a paper [4] by Gaschütz which provides the setting in which the results of Carter [1] on the existence of nilpotent self-normalizing subgroups of solvable groups take their most natural form. He showed that given a saturated formation $\mathscr{F}$, and any finite solvable group G, one can find a conjugacy class of subgroups of G (called $\mathscr{F}$-subgroups of G) which is connected in a natural way with the formation $\mathscr{F}$. Recently, Carter and Hawkes [2] have made a major contribution to the theory by generalizing the work of Philip Hall on system normalizers in solvable groups to $\mathscr{F}$-normalizers, and investigating the relationships between the $\mathscr{F}$-subgroups of a solvable group G and the $\mathscr{F}$-normalizers of G. As is clear from (1.1), this paper proceeds in a different direction by considering the relative embedding of the $\mathscr{F}$-subgroups for two distinct saturated formations $\mathscr{E}$, $\mathscr{F}$. We consider only finite solvable groups in this paper.

The machinery used in the proof of our main theorem, Theorem 5.8, is developed in § 4. We begin by obtaining a characterization of strong containment which depends only on the two formations $\mathscr{E}$ and $\mathscr{F}$. This characterization depends on the knowledge that if $\mathscr{F}$ is a saturated formation, then $\mathscr{F}$ is a locally defined formation (see