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IDEALS IN ADMISSIBLE ALGEBRAS

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The notion of admissible algebra has been introduced by Koecher. They are commutative algebras whose enveloping Lie algebra (of multiplications) splits into the direct sum of an even and an odd part. It will be shown here that the class of admissible algebras cannot be defined by (nonassociative) polynomial identities. This is done by exhibiting an admissible algebra which possesses a homomorphic image which is not admissible. The main tool is the relationship between the admissibility of a homomorphic image of an admissible algebra A, a symmetry property of a certain ideal of the enveloping Lie algebra of A formed from the kernel of the homomorphism, and the ideal structure of an algebra constructed by Koecher from A.

Let A be a commutative algebra over a field F of characteristic not two. We do not assume that A is associative, nor that A has a unit element 1. For a in A, let $L(a)$ denote multiplication by a in A, i.e., $L(a)x = ax$ for x in A. Let $L(A)$ denote the vector space of linear mappings $L(a)$, a in A. Let $H(A)$ be the Lie algebra generated by L(A). $H(A) = H^-(A) + H^+(A)$, where $H^-(A)$ is the space spanned by commutator products of an odd number of elements of $L(A)$, and $H^+(A)$ is the space spanned by commutator products of an even number of elements of $L(A)$. The commutator $[T, U]$ of T and U is $TU - UT$. If $1 \in A$, then the sum $L(A) + [L(A), L(A)]$ is direct, since commutators vanish on 1. We call A *admissible*, [1], if the identity mapping of $L(A)$ extends to an antiautomorphism $T \rightarrow T^*$ of $H(A)$. This means that $T^* = T$ for $T \in H^-(A)$, and $T^* = -T$ for $T \in H^+(A)$, so that $H^-(A) \cap H^+(A) = 0$. Conversely, if $H^-(A) \cap H^+(A) = 0$, then the map $(T + U)^* = T - U$ for $T \in H^-(A)$, $H \in H^+(A)$ shows that A is admissible.

Let A be admissible. We may then form the algebra $\mathfrak{L}(A) = H(A) \oplus A \oplus \bar{A}$, where $\bar{A}$ is a vector space copy of A. The multiplication in $\mathfrak{L}(A)$ is given by $[(T_1, a_1, \bar{b}_1), (T_2, a_2, \bar{b}_2)] = (T, a, \bar{b})$, where $T = [T_1, T_2] + a_1 \Delta \bar{b}_2 - a_2 \Delta \bar{b}_1$, $a = T_1 a_2 - T_2 a_1$, $b = T_2{}^* b_1 - T_1{}^* b_2$, and the pairing Δ of A and $\bar{A}$ into $H(A)$ is given by $a \Delta \bar{b} = L(ab) + [L(a), L(b)]$ (see [1]).

Now let M be an ideal of A. Let $j_1(M) = \{T \in H(A) \mid TA \subseteqq M\}$. $j_1(M)$ is an ideal in $H(A)$. $j_1(M)$ contains the ideal $i(M)$ of $H(A)$ generated by the mappings $L(m)$, $m \in M$. If we let $I(M) = i(M) \oplus M \oplus \bar{M}$ and $J_1(M) = j_1(M) \oplus M \oplus \bar{M}$, and note that $i(M)^* \subseteqq i(M)$, then we see that $I(M)$ is an ideal in $\mathfrak{L}(A)$. In this note, we shall give an example to show that $J_1(M)$ need not be an ideal in $\mathfrak{L}(A)$. We first