PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 1, 1969

# ELLIPTIC DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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**The purpose of this paper is to furnish a proof of the following theorem:**

**THEOREM.** $D_1$ **and** $D_2$ **are two disjoint open sets in the** $xy$**-plane having the open arc** $\sigma$ **as a common boundary.** $L_i$ **in** $D_i$, $i = 1, 2$, **are defined as**

$$L_i(\phi) \equiv a_i\phi_{xx} + 2b_i\phi_{xy} + c_i\phi_{yy} + d_i\phi_x + e_i\phi_y + g_i\phi,\ a_ic_i - b_i^2 > 0 \ .$$

**Functions** $u_i$ **satisfy** $L_i(u_i) = f_i$ **in** $D_i$, **with** $u_i \in C^2$ **in** $D_i$ **and** $\in C^1$ **in** $D_i U\sigma$; **on** $\sigma$, $u_1 = u_2$ **and** $\partial u_1/\partial N_1 = k(s)\partial u_2/\partial N_2$, **where** $s$ **is arc length on** $\sigma$, $k(s) > 0$, **and** $\partial u_i/\partial N_i$ **denotes the conormal derivative of** $u_i$. **If, on** $D_iU\sigma$, $a_i, b_i, c_i \in C_\alpha^{n+2}$; $d_i, e_i, g_i, f_i \in C_\alpha^n$; $k \in C_\alpha^{n+2}$ **and** $\sigma \in C_\alpha^{n+3}$; **then** $u_i \in C^{n+2}$ **on** $D_iU\sigma$ **for** $n \geqq 0$. **If all indicated quantities are analytic functions of their arguments and** $\sigma$ **is an analytic arc, then** $u_i$ **is analytic on** $D_iU\sigma$.

Here $u \in C_\alpha^n$ on $G$ means the $n$th order derivatives of $u$ satisfy a uniform Hölder condition with exponent $\alpha$ on every compact subset of $G$. The conormal derivative

$$\partial u_i/\partial N_i = (a_ir + b_is)u_x + (b_ir + c_is)u_y,\ r^2 + s^2 = 1 \ ,$$

uses the same unit normal $(r, s)$ to $\sigma$ for $i = 1$ or 2; this normal may point into $D_1$ or it may point into $D_2$.

Such elliptic equations occur in physical problems involving continuous media of different properties. Smoothness of solutions plays an important role in the numerical analysis of such problems [7], [6].

Oleinik [5] has investigated the smoothness of solutions to such problems in several dimensions starting with weaker differentiability hypotheses on the $u_i$. This work differs from hers in that here analyticity is proved, and no restriction on $g_i$ is required for uniqueness of solutions is not required in the proof. The proof here is also along different lines since the restriction to two dimensions allows the use of conformal mapping and the Beltrami differential equation to bring $L_i$ to normal form, allowing previous results of the author [8] to be applied.

2. Since the proof of the theorem is based on coordinate transformations it is essential to examine how the various coefficients and the problem as a whole are altered under point transformations. With the symbolism