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BEST CONSTANTS IN A CLASS OF INTEGRAL INEQUALITIES

DAVID W. BOYD

In this paper a method is developed for determining best constants in inequalities of the following form:

$$\int_a^b |y|^p |y^{(n)}|^q w(x)dx \leqq K\left\{\int_a^b |y^{(n)}|^r m(x)dx\right\}^{(p+q)/r},$$

where $y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0$ and $y^{(n-1)}$ is absolutely continuous.

It is first shown that for a certain class of m and w, equality can be attained in the inequality. Applying variational techniques reduces the determination of the best constant to a nonlinear eigenvalue problem for an integral operator. If m and w are sufficiently smooth this reduces further to a boundary value problem for a differential equation. The method is illustrated by determining the best constants in case (a, b) is a finite interval, $m(x) \equiv w(x) \equiv 1$, and $n = 1$.

A number of special cases of the inequality have been studied but usually without obtaining best constants. An exception to this is the case $n = 1, q = 0, p = r$ which was studied very thoroughly by Beesack [1], who gave a direct method for determining best constants. The method of [1] was modified by Boyd and Wong [5] to apply to the case $n = 1, q = 1, r = p + 1$. Recently Beesack and Das [2] obtained constants for the case $n = 1, r = p + q$ but these were not in general best possible.

We shall state our result only for $n = 1$ although it will be clear that the analogous result for $n > 1$ is valid. In our closing remarks we indicate a number of other inequalities to which the method of this paper applies.

1. Preliminaries. Throughout we assume that p, q, r, a, b are real numbers satisfying $p > 0, r > 1, 0 \leqq q < r$ and $-\infty \leqq a < b \leqq \infty$. The functions m and w are measurable and positive almost everywhere. We write $d\mu(x) = m(x)dx$ and

$$||f||_s = \left\{\int_a^b |f|^s d\mu\right\}^{1/s} \quad \text{for} \quad 0 < s < \infty .$$

The space L_m^s is the set of functions with $||f||_s < \infty$, with the usual identification. We shall use the notation $f_n \to f$ if $||f_n - f||_s \to 0$, and if $s \geqq 1$ so L_m^s is a Banach space, we write $f_n \xrightarrow{w} f$ for weak convergence in L_m^s. We denote the dual of L_m^s by $L_m^{s'}$ so for $s > 1$, $s' = s/(s-1)$.

We shall consider integral operators of the type