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A NOTE ON THE THEORY OF PRIMES

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In this paper we find those commutative rings for which the theory of primes is subsumed under classical ideal theory, that is, for which every finite prime is an ideal. The characterization is given in terms of domains with this property and they are shown to form a class of domains from number theory. In addition we give two characterizations of the primes of a subring of a global field. The first establishes them as the nontrivial preprimes whose complements are multiplicatively closed and the second relates the space of all primes to that of the quotient field.

The concept of a prime for commutative rings with identity was introduced by Harrison in 1966.

In what follows all rings are commutative and have a unity and all primes are finite. $X(R)$ denotes the set of primes of a ring R and $X'(R)$ denotes the set of valuation preprimes (preprimes T such that for each finite $E \subset R$, $T \cap E = \emptyset \Rightarrow$ there is $P \in X(R)$ with $T \subset P$ and $P \cap E = \emptyset$). For a preprime T of R which is closed under subtraction, define the *idealizer* $A(T)$ *of* T *in* R by $A(T) = \{a \in R : aT \subset T\}$. $A(T)$ is a subring of R in which T is an ideal.

1. Call a ring a *C-ring* if every finite prime of it is an ideal. It is easy to check that the class of C-rings is closed under taking subrings and homomorphic images.

THEOREM 1. *The following are equivalent for a ring* R:
(1) *R is a C-ring*;
(2) $X(R) = \{$*maximal ideals of* $R\}$;
(3) *R/P is a C-ring, for each minimal prime ideal P of R*;
(4) $X'(R) = \mathrm{Spec}(R)$.

Proof. That $(4) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3)$ is clear. In any case, $\mathrm{Spec}(R) \subset X'(R)$ [1, Lemma 2.6]. Let $P \in X'(R)$. P contains a minimal prime ideal Q of R and $P/Q \in X'(R/Q)$. Then P/Q is the intersection of the primes of R/Q which contain it; so, if R/Q is a C-ring, then $P/Q \in \mathrm{Spec}(R/Q)$ and $P \in \mathrm{Spec}(R)$.

Because of condition (3), we turn to the classification of C-domains. If S denotes the ring of rational integers or a ring of polynomials in one variable over a finite field, then one checks that the polynomial