PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 3, 1969

TWO CHARACTERIZATIONS OF QUASI-FROBENIUS RINGS

EDGAR A. RUTTER, JR.

The purpose of this paper is to characterize quasi-Frobenius, QF, rings in terms of relationships assumed to exist for each cyclic or finitely generated left module between the module and its second dual, where duality is with respect to the ring. More specifically we prove that a left perfect ring is QF if every cyclic left module is reflexive or every finitely generated left module is (isomorphic to) a submodule of a free module. For rings with minimum condition on left or right ideals this later condition is equivalent to every finitely generated left module being torsionless or to the ring being a cogenerator in the category of finitely generated left modules. If annihilator relations are defined by means of the natural pairing between a module and its dual, this condition is also equivalent to every submodule of a finitely generated left module being an annulet.

One of Nakayama's [14] original characterizations of QF rings was as rings with minimum condition on left or right ideals in which every left ideal is a left annulet and every right ideal is a right annulet. Ikeda and Nakayama [7] proved that a finite dimensional algebra in which every left ideal is a left annulet is QF but Nakayama [15] gave an example which shows that this weaker assumption does not characterize QF rings even when one assumes minimum condition on both left and right ideals. For an arbitrary ring every left ideal being a left annulet is easily seen to be equivalent to every cyclic left module being torsionless. Thus our results may be regarded as an attempt to extend the one sided result of Ikeda and Nakayama mentioned above by making weaker finiteness assumptions and stronger assumptions about the relation of the members of various classes of left modules to their second duals. Our results are also related to those of Morita, Kawada and Tachikawa [12] who proved that a ring with minimum condition is QF if every left module is a submodule of a free module and of Faith and Walker [5] who recently showed that this characterization is still valid without the assumption of minimum condition. Finally we call attention to the investigations of B.L. Osofsky [17] which we cannot adequately relate to the above results in a few words.

1. Preliminaries. Let M and N be modules over a ring R. The bimodule character of R induces an R-module structure on $M^* =$