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ESTIMATES OF POSITIVE CONTRACTIONS

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The purpose of this paper is to obtain an L_p estimate for the supremum of the Cesàro averages of a certain class of positive contractions of L_p. Let $(X, \mathscr{F}, \mu)$ be a measure space, and let T be a linear operator mapping $L_p(X, \mathscr{F}, \mu)$ into itself for p fixed, $1 < p < +\infty$. If there is a constant $c > 0$ such that for each $f \in L_p(X, \mathscr{F}, \mu)$,

$$\int \sup_n |f, (f + Tf)/2, \cdots, (f + Tf + \cdots + T^n f)/n + 1|^p d\mu \leqq c^p \int |f|^p d\mu ,$$

then we say that T admits of a dominated estimate with constant c. In an effort to unify certain results due to A. Ionescu-Tulcea and to E. Stein, a somewhat more general form of the following theorem was obtained earlier: If T is a positive contraction, and if there exists an $h > 0$ a.e., $h \in L_p(X, \mathscr{F}, \mu)$ and $Th = h$, then T admits of a dominated estimate with constant $p/p - 1$. In the present paper, we have extended the theorem, obtaining a slightly more general form of the following: If T is a positive contraction and if for each positive integer n there exists an $h_n > 0$ a.e., $h_n \in L_p(X, \mathscr{F}, \mu)$ and $\|h_n\| = \|T^n h_n\|$, then T admits of a dominated estimate with constant $p/p - 1$.

This result is more widely applicable more directly than the previous theorem, but is not the most general result one might conjecture, that positive contractions admit of a dominated estimate with no further assumptions. In this direction, we have obtained several equivalent formulations of the problem which may help to lead to an answer. In any case, it remains an open problem whether or not positive contractions of $L_p(X, \mathscr{F}, \mu)$, $1 < p < +\infty$, admit of a dominated estimate without the assumption of additional conditions.

2. Main results. Let $(X_1, \mathscr{F}_1, \mu_1)$ and $(X_2, \mathscr{F}_2, \mu_2)$ be two measure spaces and let T be a linear operator mapping $L_p(X_1, \mathscr{F}_1, \mu_1)$ into $L_p(X_2, \mathscr{F}_2, \mu_2)$, p fixed, $1 \leqq p \leqq +\infty$. We say that T is a contraction if its norm is less than or equal to one. We say that T is positive if it maps nonnegative functions to nonnegative functions. We shall omit the phrase almost everywhere, it being understood where applicable.

DEFINITION 2.1. The range set of T, $R(T)$, is the support of Tf,