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DECOMPOSITIONS OF INJECTIVE MODULES

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The main results of this paper concern decompositions of an injective module, either as a direct sum of submodules or as the injective envelope of a direct sum of injective submodules. This second kind of decomposition can be regarded as an ordinary direct sum (coproduct) in a suitable Abelian category—the spectral category of the ring. The results are therefore put in the context of Abelian categories, and the main result is that in an Abelian category satisfying axiom Ab-5 and with infinite direct sums, any two direct sum decompositions of an injective object have isomorphic refinements.

This is particularly strong if decompositions into indecomposable injectives exist, and it enables one to classify the injective modules over a valuation ring. Such strong results as this are not available for more general classes of modules, but in § 3 the methods of Crawley and Jónsson are exploited to obtain results in certain cases; for example, for modules which are direct sums of countably generated modules. The Crawley-Jónsson results are put into the context of category theory and an example is given (involving relatively injective modules) to show how the hypotheses can be weakened by working in a subcategory of the category of R-modules.

A remark should be made on the types of decompositions we consider for injective modules in § 2. For injective modules over Noetherian rings, ordinary direct sums yield excellent results, due primarily to Matlis [7]. In contrast, Faith and Walker [2] have shown that if R is a non-Noetherian ring, there does not exist any set of injective modules such that any injective module can be imbedded in a direct sum of modules isomorphic to members of this set. In the spectral category, however, reasonable decompositions always exist (Theorem 2 below). The spectral category was introduced by Gabriel and Oberst in [4] and exploited in [10]. The author is indebted to Professor J. E. Roos for pointing out the connection between these two papers and the work reported here.

We do not consider Cartesian product decompositions of injective modules, since product decompositions simply do not have the necessary uniqueness properties. For an example let Q and Z denote the additive groups of rationals and integers, respectively, and $(Q/Z)_p$ the p-primary component of Q/Z. Then

$$\prod_p (Q/Z)_p \cong Q \times \prod_p (Q/Z)_p$$

so that we have two product decompositions of an injective Z-module