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EXISTENCE OF A SPECTRUM FOR NONLINEAR TRANSFORMATIONS

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Denote by S a complex (nondegenerate) Banach space. Suppose that T is a transformation from a subset of S to S. A complex number λ is said to be in the resolvent of T if $(\lambda I - T)^{-1}$ exists, has domain S and is Fréchet differentiable (i.e., if p is in S there is a unique continuous linear transformation $F = [(\lambda I - T)^{-1}]'(p)$ from S to S so that

$$\lim_{q \to p} || q - p ||^{-1} || (\lambda I - T)^{-1} q - (\lambda I - T)^{-1} p - F(q - p) || = 0)$$

and locally Lipschitzean everywhere on S. A complex number is said to be in the spectrum of T if it is not in the resolvent of T.

Suppose in addition that the domain of T contains an open subset of S on which T is Lipschitzean.

THEOREM. *T has a (nonempty) spectrum.*

If T is a continuous linear transformation from S to S, then the notion of resolvent and spectrum given here coincides with the usual one ([1], p. 209, for example). Such a transformation T is, of course, Lipschitzean on all of S and hence the above theorem gives as a corollary the familiar result that a continuous linear transformation on a complex Banach space has a spectrum.

The set of all complex numbers is denoted by C.

LEMMA. *Suppose that $d > 0$, p is in S, Q is a transformation from a subset of S to S, D is an open set containing p which is a subset of the domain Q, Q is Lipschitzean on D and $(I - cQ)^{-1}$ exists and has domain S if c is in C and $|c| < d$. Then,*

$$\lim_{c \to 0} (I - cQ)^{-1} p = p .$$

Proof. Denote by M a positive number so that $|| Qr - Qs || \leqq M || r - s ||$ if r and s are in D. Suppose $\varepsilon > 0$. Denote by δ a number so that $0 < \delta < \min(\varepsilon, 1/2)$ and $\{q \in S: || q - p || \leqq \delta\}$ is a subset of D. Denote by δ' a positive number so that $\delta'(\max(M, || Qp ||)) < \delta/2$. Denote by c a member of C so that $|c| < \min(\delta', d)$. Denote $(I - cQ)^{-1}p$ by q, denote p by q_0 and $p + cQq_{n-1}$ by q_n, $n = 1, 2, \cdots$.

Then, $|| q_1 - q_0 || = || p + cQq_0 - q_0 || = |c| \, || Qq_0 || < \delta/2$. Suppose that k is a positive integer so that

$$|| q_m - q_{m-1} || < (\delta/2)^m, m = 1, 2, \cdots, k .$$