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TENSOR PRODUCTS OF COMPACT CONVEX SETS

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Suppose that K_1 and K_2 are compact convex subsets of locally convex spaces E_1 and E_2 respectively. There are several definitions of new compact convex sets associated with K_1 and K_2, each of which may reasonably be called a "tensor product" of K_1 and K_2. We compare these different tensor products and their extreme points; in doing so, we obtain some new characterizations of Choquet simplexes, another formulation of Grothendieck's approximation problem and much simpler proofs of known characterizations of the extreme points of these tensor products. Most of these results are obtained as special cases of theorems in the first half of the paper which deal with the state spaces of tensor products of partially ordered linear spaces with order unit.

1. Tensor products of partially ordered spaces. *A partially ordered linear space with order unit* is a triple (E, P, u), where the linear space E is given the partial ordering induced by the cone P, where $P \cap (-P) = \{0\}$, and where u is an order unit for P, i.e., $P - u$ absorbs E. Given a partially ordered linear space (E, P), the *dual cone* $P^\star$ is the space of all linear functionals on E which are nonnegative on P. The subspace of the algebraic dual of E which is generated by $P^\star$ is denoted by $E^\star$; it is clear that $E^\star = P^\star - P^\star$. The partially ordered linear space $(E^\star, P^\star)$ is called the *order dual* of (E, P).

If (E_1, P_1, u_1) and (E_2, P_2, u_2) are two partially ordered linear spaces with order units, then in the tensor product $E_1 \otimes E_2$ the cone generated by elements of the form $x_1 \otimes x_2$ $(x_i \in P_i)$ will be denoted by $P_1 \otimes P_2$. The triple $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ is a partially ordered linear space with order unit [3, 8].

Given a partially ordered linear space with order unit (E, P, u), its *state space* S is the set of all f in $P^\star$ such that $\langle f, u\rangle = 1$, provided with the weak* $(=w(E^\star, E))$ topology. Clearly, S is convex compact and Hausdorff. It is possible for S to be empty (cf. [7, p. 26]). If S is the state space of $(E_1 \otimes E_2, P_1 \otimes P_2, u_1 \otimes u_2)$ and $s \in S$, then there exists a related functional s_1 on E_1 defined by

$$\langle s_1, x_1\rangle = \langle s, x_1 \otimes u_2\rangle , \qquad x_1 \in E_1 .$$

It is clear that s_1 is in the state space S_1 of (E_1, P_1, u_1), and it is clear how to define the analogous state s_2 in S_2. In the reverse direction, suppose that t_i is in the state space S_i of (E_i, P_i, u_i) and define the functional $t_1 \otimes t_2$ on $E_1 \otimes E_2$ by setting