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ON COMMUTATIVE RINGS OVER WHICH THE SINGULAR SUBMODULE IS A DIRECT SUMMAND FOR EVERY MODULE

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A commutative ring R with 1 over which the singular submodule is a direct summand for every module, is a semi-hereditary ring with finitely many large ideals. A commutative semi-simple (with d.c.c.) ring is characterized by the property that every semi-simple module is injective.

In this note we continue our investigation of the commutative non-singular rings over which for any module M, the singular submodule $Z(M)$ is a direct summand [1]. As in [1] we say that such a ring has SP. Throughout this paper a ring R is commutative with identity 1 and all modules are unitary. A ring is regular (in the sense of Von Neumann) if every finitely generated right ideal of R is generated by an idempotent; semi-simple means semi-simple with d.c.c.. Notation and terminology here is as in [1].

In [1] we established the following characterization of rings with SP, included here for easy reference:

THEOREM 1. *For a ring R the following are equivalent:*

(a) *R has SP*

(b) *R is regular and has BSP* [1].

(c) *$Z(R) = (0)$ and for every large ideal I or R, the ring R/I is semi-simple.*

(d) *Every R-module M with $Z(M) = M$ is R-injective.*

In particular if R has SP, then R is hereditary.

We shall need the following corollaries of this theorem:

COROLLARY 1.1. *Every homomorphic image of a ring R with SP, has SP.*

Proof. Let $S = R/I$ for some ideal $I(\neq (0), R)$ of R; S is regular since R is and thus $Z(S) = (0)$. A large ideal A of S is of the form J/I where J is a large ideal of R containing I. Thus it follows from (c) that S/A is semi-simple since $S/A \cong R/J$. Now S has SP since it satisfies (c).

COROLLARY 1.2. *Every singular module over a ring R with SP*