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ON $(\mathfrak{m} - \mathfrak{n})$ PRODUCTS OF BOOLEAN ALGEBRAS

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This discussion begins with the problem of whether or not all $(\mathfrak{m} - \mathfrak{n})$ products of an indexed set $\{\mathfrak{A}_t\}_{t \in T}$ of Boolean algebras can be obtained as $\mathfrak{m}$-extensions of a particular algebra $\mathscr{F}_{\mathfrak{n}}^*$. The construction of $\mathscr{F}_{\mathfrak{n}}^*$ is similar to the construction of the Boolean product of $\{\mathfrak{A}_t\}_{t \in T}$; however the $\mathscr{A}_t$ are embedded in $\mathscr{F}_{\mathfrak{n}}^*$ in such a way that their images are $\mathfrak{n}$-independent. If there is a cardinal number $\mathfrak{n}'$, satisfying $\mathfrak{n} < \mathfrak{n}' \leqq \mathfrak{m}$, then $(\mathfrak{m} - \mathfrak{n}')$ products are not obtainable in this manner. For the case $\mathfrak{n} = \mathfrak{m}$ an example shows the answer to be negative. It is explained how the class of $\mathfrak{m}$-extensions of $\mathscr{F}_{\mathfrak{n}}^*$ is situated in the class of all $(\mathfrak{m} - \mathfrak{n})$ products of $\{\mathfrak{A}_t\}_{t \in T}$. A set of $\mathfrak{m}$-representable Boolean algebras is given for which the minimal $(\mathfrak{m} - \mathfrak{n})$ product is not $\mathfrak{m}$-representable and for which there is no smallest $(\mathfrak{m} - \mathfrak{n})$ product.

These problems have been proposed by R. Sikorski (see [2]). Concerning $\{\mathfrak{A}_t\}_{t \in T}$, it is assumed throughout that each of these algebras has at least four elements. $\mathfrak{m}$ and $\mathfrak{n}$ will always denote infinite cardinals with $\mathfrak{n} \leqq \mathfrak{m}$. All definitions are taken from [2]. An $\mathfrak{m}$-homomorphism is a homomorphism that is conditionally $\mathfrak{m}$-complete. We denote the class of $(\mathfrak{m} - \mathfrak{n})$ products of $\{\mathfrak{A}_t\}_{t \in T}$ by $\boldsymbol{P}_{\mathfrak{n}}$ and the class of $(\mathfrak{m} - 0)$ products by $\boldsymbol{P}$. Let $\{\{i_t\}_{t \in T}, \mathscr{B}\}$ and $\{\{j_t\}_{t \in T}, \mathfrak{C}\}$ be elements of $\boldsymbol{P}$. We say that

$$\{\{i_t\}_{t \in T}, \mathscr{B}\} \leqq \{\{j_t\}_{t \in T}, \mathfrak{C}\}$$

provided there is an $\mathfrak{m}$-homomorphism h from $\mathfrak{C}$ onto $\mathscr{B}$ such that $h \circ j_t = i_t$ for $t \in T$. The relation "$\leqq$" is a quasi-ordering of $\boldsymbol{P}$. Two $(\mathfrak{m} - 0)$ products are isomorphic if each is $\leqq$ to the other.

The particular product, $\{\{g_t^*\}_{t \in T}, \mathscr{F}_{\mathfrak{n}}^*\}$ of $\{\mathfrak{A}_t\}_{t \in T}$ mentioned above is defined as follows. For each $t \in T$ let X_t be the Stone space of $\mathfrak{A}_t$ and let g_t be an isomorphism from $\mathfrak{A}_t$ onto the field $\mathscr{F}_t$ of all open and closed subsets of X_t. Let X be the Cartesian product of the sets X_t, and for each $t \in T$ and each $b \in \mathfrak{A}_t$, set

$$g_t^*(b) = [x \in X: x(t) \in g_t(b)\} . \tag{1}$$

Let $G_{\mathfrak{n}}$ be the set of all subsets a of X which satisfy the following condition:

$$a = \bigcap_{t \in S} g_t^*(b_t) \text{ where } b_t \in \mathfrak{A}_t,\ S \subseteqq T \text{ and } \bar{\bar{S}} \leqq \mathfrak{n} .$$

Finally, let $\mathscr{F}_{\mathfrak{n}}^*$ be the field of subsets of X which is generated by $G_{\mathfrak{n}}$.