PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 3, 1969

ON A CALCULUS OF PARTITION FUNCTIONS

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The main object in this paper is to show that many partition theorems which have been deduced from identities in basic hypergeometric series and infinite products may in fact be given purely combinatorial proofs. We show that the manipulations performed on the generating functions have combinatorial interpretations, and thus we obtain a "calculus of partition functions" which translates a sizable portion of the techniques of the elementary theory of basic hypergeometric series into arithmetic terms.

In [13], Vahlen derived a large number of partition theorems combinatorially. Some of his initial results are actually arithmetic proofs of simple infinite product identities. For example, his derivation of equation (10) [13; p. 4] is an arithmetic proof of

$$\frac{\prod_{j=1}^{\infty}(1-q^j)}{\prod_{h=1}^{\infty}(1-q^h)} = 1 .$$

In §2, we shall extend the results of Vahlen (Lemmas 3, 4, and 5) and derive some further arithmetic proofs of well-known identities. These results will form the basis of our calculus. In §3, we illustrate the use of our calculus by giving new combinatorial proofs of Euler's theorem [9; p. 277, Th. 344] and Jacobi's identity [9; p. 282].

It should be stressed that the interest of these results lies not so much in their contribution to the search for new partition theorems as in their clarification of the relationship between combinatorial partition theory and what was previously the purely analytic aspect of partition theory. Thus we include in §3 only two results of a rather simple character; even these are relatively complicated to prove by our calculus. However, the method of proof is equally applicable to all the analytic results in [2], [3], [4], [5]; in such results Lemma 2 is crucial. And indeed the result on the order of a partition in [8] indicates that one of Rogers's proofs of the Rogers-Ramanujan identities may now be translated into a combinatorial proof.

2. Fundamental lemmas. We let $\sum$ denote the set of all doubly infinite sequences of nonnegative integers $\{f_n\}_{n=-\infty}^{\infty} = \{f_n\}$ for which $f_n = 0$ for all but finitely many n. We define a *partition condition* R to be a subset of $\sum$. We say that a partition n of the