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A CONJECTURE AND SOME PROBLEMS ON PERMANENTS

G. N. DE OLIVEIRA

Let $A = [a_{ij}]$ denote an $n \times n$ matrix and let E be the $n \times n$ identity matrix. We will designate by det A and perm A the determinant and the permanent of A respectively. The polynomial $\varphi(z) = \det(zE - A)$ plays a fundamental role in matrix theory. Similarly we can consider the polynomial $f(z) = \text{perm}\,(zE - A)$ which has been object of several studies recently, particularly when A is a doubly stochastic matrix. The aim of the present paper is to give some results on the existence of matrices satisfying certain conditions involving the roots of this polynomial.

Let M_n and $\mathscr{M}_n$ be the regions defined as follows: $z \in M_n$ if and only if there exists a stochastic matrix of order n with z as characteristic root; $(z_1, \cdots, z_n) \in \mathscr{M}_n$ if and only if there exists a stochastic matrix of order n whose n characteristic roots are the complex numbers $z_1, \cdots, z_n$.

Similarly we define the regions D_n and $\mathscr{D}_n$ respectively when 'stochastic' is replaced by 'doubly stochastic'. M_n was determined by Karpelevič [3] but the determination of the other three regions seems to be a very difficult problem and has not yet been solved (see [7], [8], [9]).

Replacing in the definitions of M_n, $\mathscr{M}_n$, D_n and $\mathscr{D}_n$ 'characteristic root' by 'root of the polynomial $f(z) = \text{perm}\,(zE - A)$' we can define four other regions which we shall denote by M_n^*, $\mathscr{M}_n^*$, D_n^* and $\mathscr{D}_n^*$ respectively. To our knowledge no attempt has been made to determine these regions. Their determination is likely to be a much harder problem than the determination of M_n, $\mathscr{M}_n$, D_n and $\mathscr{D}_n$.

Some problems dealing with the characteristic values of a matrix (like some of the problems mentioned in [6]) can be replaced by similar problems dealing with the roots of

$$f(z) = \text{perm}\,(zE - A)\ .$$

Examples: (1) find a necessary and sufficient condition for the numbers $a_1, \cdots, a_n$ and $z_1, \cdots, z_n$ to be the principal elements of a symmetric A and the roots of $f(z) = \text{perm}\,(zE - A)$ respectively; (2) find a necessary and sufficient condition for the numbers $\lambda_1, \cdots, \lambda_n$ and $z_1, \cdots, z_n$ to be the characteristic roots of an $n \times n$ matrix A and the roots of $f(z) = \text{perm}\,(zE - A)$ respectively. In the sequel we give some results on problems of this nature.

2. Let