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ON NORMED RINGS WITH MONOTONE MULTIPLICATION

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It is shown that if a normed division ring has a norm which is "multiplication monotone" in the sense that $N(x) < N(x')$ and $N(y) < N(y')$ imply $N(xy) \leqq N(x'y')$, and if the norm is "commutative" in the sense that $N(\cdots xy \cdots) = N(\cdots yx \cdots)$ for all x and y, then the topology of that ring is given by an absolute value. A consequence of this result is that if the norm of a connected normed ring with unity is multiplication monotone and commutative then the ring is embeddable in the system of quaternions.

Pontrjagin has shown [7] that the only locally compact connected fields are the field of real numbers and the field of complex numbers. A theorem of A. Ostrowski [6] implies that if the topology of a connected field is given by an absolute value then the field is (isomorphic to) a subfield of the field of complex numbers. Both results are contributions toward the solution of the problem of determining what connected fields exist.

In this note the more restricted question of studying connected normed fields is considered. (It is recalled that a *normed ring* has its topology induced by a *norm* function N; that is, N is a real-valued function defined on the ring such that: (i) $N(0) = 0$ and $N(x) > 0$ for $x \neq 0$, (ii) $N(-x) = N(x)$ for all x, (iii) $N(x + y) \leqq N(x) + N(y)$ for all x and y, (iv) $N(xy) \leqq N(x)N(y)$ for all x and y.) Ostrowski's results may be regarded as the treatment of the special case of this problem in which the norm N satisfies the additional condition

$$N(xy) = N(x)N(y)$$

for all x and y. This extra requirement is replaced here by the weaker condition that N be *multiplication monotone* in the sense that whenever $N(x) < N(x')$ and $N(y) < N(y')$ then $N(xy) \leqq N(x'y')$.

Specifically, it is shown in the corollary of Theorem 3 that if a commutative connected normed ring with unity has a multiplication monotone norm then that ring is (algebraically and topologically isomorphic to) a subring of the field of complex numbers. (The version of this statement which appears below actually includes the noncommutative case as well.) The basic device employed in obtaining this result is Theorem 2, which asserts that if a normed division ring has a multiplication monotone norm N such that

$$N(\cdots xy \cdots) = N(\cdots yx \cdots)$$