PACIFIC JOURNAL OF MATHEMATICS
Vol. 33, No. 2, 1970

THE COMMUTATOR AND SOLVABILITY IN A GENERALIZED ORTHOMODULAR LATTICE

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In this paper we prove in a generalized orthomodular lattice the analog of the following theorem from group theory. For a and b members of a group G, let $aba^{-1}b^{-1}$ be the commutator of a and b. The set of commutators in G generates a normal subgroup H of G possessing these properties: G/H is Abelian. Moreover, if K is any normal subgroup of G for which G/K is Abelian, then $K \supseteqq H$. Continuing the analogy with group theory, we determine a solvability condition on generalized orthomodular lattices.

An *orhomodular lattice* is a lattice L with 0 and 1 and with an orthocomplementation $'$: $L \rightarrow L$ satisfying the *orthomodular identity*: for $e \leqq f$ in L, $f = e \vee (f \wedge e')$. Throughout this paper L shall denote an orthomodular lattice. For $f \in L$ the *Sasaki projection determined by* f $\phi_f : L \rightarrow L$ by $e\phi_f = (e \vee f') \wedge f$. We say e *commutes with* f, ecf, when $e\phi_f = e \wedge f$. Basic properties of orthomodular lattices and of their coordinatizing Baer $*$-semigroups are contained in [1, 2].

A lattice ideal I in L is called a *p-ideal* if and only if $e \in I$ and $f \in L$ imply $e\phi_f \in I$. Theorem 6, which concerns p-ideals in generalized orthomodular lattices, indicates the significance of p-ideals in orthomodular lattices.

2. The commutator.

For elements e and f of the orthomodular lattice L, we define the *commutator* of e and f by

$$[e, f] = (e \vee f) \wedge (e \vee f') \wedge (e' \vee f) \wedge (e' \vee f') .$$

It is easily shown that ecf if and only if $[e, f] = 0$, and that $[e, f] = [e, f'] = [e', f] = [e', f']$.

THEOREM 1. *Let R be a Baer $*$-ring, and let $P'(R)$ denote the orthomodular lattice of closed projections in R. Then for*

$$e, f \in P'(R), (ef\text{-}fe)'' = [e, f] .$$

In proving the theorem, we shall use the following computation.

LEMMA 2. $[e, f] = (f'ef)'' \vee (e'fe)''$.

Proof. $(f'ef)'' = ((f'e)''f)'' = f'\phi_e\phi_f = \{[(f' \vee e') \wedge e] \vee f'\} \wedge f =$