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SECTIONS AND SUBSETS OF SIMPLEXES

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There is a locally convex space E and a compact simplex $S \subset E$ with the following property: for any metrizable compact convex subset K of a locally convex space there is a subspace $M \subseteq E$ such that K is affinely homeomorphic to $M \cap S$. One possible choice is $E = l_1$ with the w^* topology induced by c and

$$S = \left\{(a_n) \colon \sum_{n=1}^{n} a_n = 1, a_n \geqq 0, n = 1, 2, \cdots\right\}.$$

If X is a Banach space and $S \subset X$ is a compact simplex, then for each $\varepsilon > 0$ there is an operator $T \colon X \to X$ with finite dimensional range such that $\| T(x) - x \| < \varepsilon$ for all $x \in S$. Every infinite dimensional Banach space X contains a compact set K for which there is no bounded simplex $S \subset X$ with $K \subset S$.

In a recent paper Phelps [12] gave an example of a three dimensional section of a compact simplex which is not a convex polytope. Our first result shows that this is not an exception: each metrizable compact convex subset of a locally convex space can be represented as the intersection of a certain compact simplex with a suitable linear space. In §3 we pass from sections of simplexes to compact simplexes in Banach spaces. The interest in their investigation is motivated by the fact that the identity operator of a Banach space can be approximated on a compact simplex of the space by operators having finite dimensional ranges (Theorem 3). Obviously the same property is shared by any set contained in a compact simplex of a Banach space. One says that a Banach space X has the approximation property [6, p. 165] if for every compact $K \subset X$ and every $\varepsilon > 0$ there is a bounded linear operator $T \colon X \to X$ with finite dimensional range such that $\| T(x) - x \| < \varepsilon$ for any $x \in K$. It is an open problem if every Banach space has the approximation property. Theorem 3 states that such an operator T can always be found when K is a subset of a compact simplex $S \subseteqq X$. However our result is probably very far from leading to an affirmative solution of this problem of Grothendieck. Indeed, in §4 we prove that every infinite dimensional Banach space contains a compact set which is too large to be included in a bounded simplex of the space (Corollary 6). At the end we mention some open problems related to the material contained in this paper.

1. We shall consider only linear spaces over the real field R.