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EXTREMELY AMENABLE ALGEBRAS

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Let S be a semigroup and $m(S)$ the space of bounded real functions on S. A subalgebra of $m(S)$ is extremely left amenable (ELA) if it is (sup) norm closed, left translation invariant, containing constants and has a multiplicative left invariant mean. S is ELA if $m(S)$ is ELA. In this paper, we give a method in constructing all ELA subalgebras of $m(S)$; it turns out that any such subalgebra of $m(S)$ is contained in an ELA subalgebra which is the uniform limit of certain classes of simple functions on S.

A subset $E \subseteqq S$ is left thick if for any finite subset $\sigma \subseteqq S$, there exists $s \in S$ such that $\{as;\ a \in \sigma\} \subseteqq E$. In §3, we strengthen a result of T. Mitchell and prove that a semigroup S is ELA if and only if for any subset $E \subseteqq S$, either E is left thick or $S - E$ is left thick. We also show how this result may be generalized to certain subalgebras of $m(S)$.

ELA semigroups and subalgebras have been considered by Mitchell in [9] and [10], and Granirer in [5], [6] and [7]. ELA semigroups S are shown to be characterized by the fixed point property on compact hausdorff spaces by Mitchell [9] and by the algebraic property: "for any a, b in S, there is a c in S such that $ac = bc = c$" by Granirer [5]. ELA subalgebras are characterized by Mitchell [10] by a fixed point property on compacta (under certain kinds of actions of S on a compact hausdorff space).

1. Some notations and preliminaries.

Let S be a semigroup. For each $a \in S, f \in m(S)$, denote by the sup norm of f, $\|f\| = \sup_{s \in S} |f(s)|$ (and it is only this norm that will be used throughout this paper), ${}_af(s) = f(as)$ and $p_a(f) = f(a)$ for all $s \in S$. Then p_a is called the *point measure* on $m(S)$ at a and any element in $\mathrm{Co}\{p_a;\ a \in S\}$ is called a *finite mean* on $m(S)$ (where $\mathrm{Co}\,A$ denotes the convex hull of a subset A in a linear space).

If A is a norm closed left translation invariant subalgebra of $m(S)$ (i.e., ${}_af \in A$ whenever $f \in A$ and $a \in S$) containing 1, the constant one function on S, and $\varphi \in A^*$, then φ is a *mean* if $\varphi(f) \geqq 0$ for $f \geqq 0$, and $\varphi(1) = 1$; φ is *multiplicative* if $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in A$; φ is *left invariant* if $\varphi({}_sf) = \varphi(f)$ for all $s \in S$ and $f \in A$; and φ is a *point measure* [*finite mean*] on A if φ is the restriction of some point measure [finite mean] on $m(S)$ to A. It is well-known that the set of [point measure] finite mean on A is w^*-dense (i.e., $\sigma(A^*, A)$