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ON EMBEDDINGS OF 1-DIMENSIONAL COMPACTA IN A HYPERPLANE IN E^4

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In this note a proof of the following theorem is given.

THEOREM 1. Suppose that X is a 1-dimensional compactum in a 3-dimensional hyperplane E^3 in euclidean 4-space E^4, that $\varepsilon > 0$, and that $f: X \rightarrow E^3$ is an embedding such that $d(x, f(x)) < \varepsilon$ for each $x \in X$. Then there exists an ε-push h of (E^4, X) such that $h|X = f$.

The proof of Theorem 1 is based on a technique exploited by the first author in [3]. This method requires that one be able to push X off of the 2-skeleton of an arbitrary triangulation of E^4 using a small push of E^4. This could be done very easily if it were possible to push X off of the 1-skeleton of a given triangulation of E^3 via a small push of E^3. Unfortunately, this cannot be accomplished unless X has some additional property (such as local contractibility) as demonstrated by the examples of Bothe [2] and McMillan and Row [9]. However, we are able to overcome this difficulty by using a property of twisted spun knots obtained by Zeeman [10].

In the following theorem let B^4 denote the unit ball in E^4, B^3 the intersection of B^4 with the 3-plane $x_4 = 0$, and D^2 the intersection of B^4 with the 2-plane $x_1 = x_2 = 0$.

THEOREM 2. *Let X be a 1-dimensional compactum in B^3 such that $X \cap \operatorname{Bd} D^2 = \emptyset$. Then there exists an isotopy $h_t: B^4 \rightarrow B^4$ ($t \in [0, 1]$) such that*

(i) *$h_0 =$ identity,*
(ii) *$h_t | \operatorname{Bd} B^4 =$ identity for each $t \in [0, 1]$, and*
(iii) *$h_1(X) \cap D^2 = \emptyset$.*

Proof. Let $I = D^2 \cap B^3$. Since X does not separate B^3, there exists a polygonal arc J in $B^3 - X$ joining one endpoint of I to the other. We may assume, by applying an appropriate isotopy of B^4, that J_+, the intersection of J with the half-space $x_3 \geqq 0$ is contained in I. Let F be a 3-cell in B^3 such that $F \cap J = J_+$ and $F \cap X = \emptyset$, and let J_- be the intersection of J with the half-space $x_3 \leqq 0$. Now spin the arc J_- about the plane $x_3 = x_4 = 0$, twisting once, so that at time $t = \pi$, J_- lies in F. (See Zeeman [10] for the details of this construction.) Observe that the boundary of the 2-cell C traced out by J_- is the same as $\operatorname{Bd} D^2$.