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FRATTINI SUBALGEBRAS OF A CLASS OF SOLVABLE LIE ALGEBRAS

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In this paper the Lie algebra analogues to groups with property E of Bechtell are investigated. Let $\mathfrak{X}$ be the class of solvable Lie algebras with the following property: if H is a subalgebra of L, then $\phi(H) \subseteqq \phi(L)$ where $\phi(L)$ denotes the Frattini subalgebra of L; that is, $\phi(L)$ is the intersection of all maximal subalgebras of L. Groups with the analogous property are called E-groups by Bechtell. The class $\mathfrak{X}$ is shown to contain all solvable Lie algebras whose derived algebra is nilpotent. Necessary conditions are found such that an ideal N of $L \in \mathfrak{X}$ be the Frattini subalgebra of L. Only solvable Lie algebras of finite dimension are considered here.

The following notation will be used. We let $N(L)$ be the nil radical of L and $S(L)$ be the socle of L; that is, $S(L)$ is the union of all minimal ideals of L. If A and B are subalgebras of L, let $Z_B(A)$ be the centralizer of A in B. The center of A will be denoted by $Z(A)$. If $[B, A] \subseteqq A$, we let $\mathrm{Ad}_A(B) = \{\mathrm{ad}\, b$ restricted to A; for all $b \in B\}$. L' will be the derived algebra of L and $L'' = (L')'$.

PROPOSITION 1. *Let L be a Lie algebra such that L' is nilpotent. Then the following are equivalent*:

(1) $\phi(L) = 0$.

(2) $N(L) = S(L)$ *and* $N(L)$ *is complemented by a subalgebra.*

(3) L' *is abelian, is a semi-simple L-module and is complemented by a subalgebra.*

Under these conditions, Cartan subalgebras of L are exactly those subalgebras complementary to L'.

Proof. Assume (1) holds. Nilpotency of L' implies $\phi(L) \supseteqq L''$, so L' is abelian and may be regarded as an L/L'-module. We may assume $L' = \sum \oplus V_\rho$, V_ρ indecomposable L/L'-submodules. If M is a maximal subalgebra of L and if $V_\rho \not\subseteqq M$, then $M \cap V_\rho$ is an ideal of L. If S is an L/L'-submodule of V_ρ properly contained between $M \cap V_\rho$ and V_ρ, then $M + S$ is a subalgebra of L properly contained between M and L, contradicting the maximality of M. Therefore M contains all maximal submodules of V_ρ for each ρ. Then $\phi(L) = 0$ implies the intersection of all maximal submodules of V_ρ is zero for each ρ. If $V_1, \cdots, V_s$ are maximal submodules of V_ρ with $V_1 \cap \cdots \cap V_s = 0$ and are minimal with respect to this property, we have $V = V_2 \cap \cdots V_s \neq 0$