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A NOTE ON HANF NUMBERS

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We show that for every $\xi < (2^\kappa)^+$, there is a theory T and set of types P in a language of power κ, such that there is a model of T which omits every $p \in P$ of power λ if and only if $\lambda \leqq \beth_\xi$. We also disprove a conjecture of Morley on the existence of algebraic elements.

The results which are proved here appear in [5].

1. On η_κ.

DEFINITION 1.1. η_κ will be the first cardinal such that for every language L, $|L| \leqq \kappa$, and set of types $\{p: p \in P\}$ (in L) if T has a model of power $\geqq \eta_\kappa$ which omits all the types in P, then T has such models in every power $\geqq |T|$. (A type is a set of formulas with the variables $x_0, \cdots, x_n$ only for some $n < \omega$. A model omits p if there does not exist $a_0, \cdots, a_n$ in the model such that $\varphi(x_0, \cdots, x_n) \in p$ implies $M \models \varphi[a_0, \cdots, a_n]$.)

Chang showed in [2], by methods of Morley from [4] that $\eta_\kappa \leqq \beth[(2^{|T|})^+]$. He also in [1] asked what is η_κ. We shall show that $\eta_\kappa = \beth[(2^\kappa)^+]$. For this it is sufficient to prove that for every $\xi < (2^\kappa)^+$ there exists a theory T and a set of types P (in a language $L = L(T)$ of power $\leqq \kappa$) such that T has a model of power λ which omits all the types in P if and only if $\lambda \leqq \beth_\xi$.

The following theorem appears in many articles which deals with finding lower bounds for Hanf numbers.

THEOREM 1.1. *If there exists a theory T, $|L(T)| \leqq \kappa$, and a set of types P in $L(T)$, such that every model of T which omits every $p \in P$ is well ordered in an order type $\leqq \xi$, and it has such a model whose order type is ξ, then $\eta_\kappa > \beth_\xi$.*

Proof. We adjoin to L the predicates $Q_1(x)$, $Q(x)$, $x \in y$, the constants c_n, $n < \omega$ and the function $F(x)$, and we get a language L_1, $|L_1| \leqq \kappa$. We define $T_1 = \{\psi^Q: \psi \in T\}$ [ψ^Q is ψ relativized to Q, that is instead of $(\exists x)\varphi$ we write $(\exists x)(Q(x) \wedge \varphi)$ and instead of $(\forall x)\varphi$ we write $(\forall x)[(Q(x) \rightarrow \varphi)]$. We also define $P_1 = \{p^Q: p \in P\} \cup \{q\}$, $p^Q = \{\varphi^Q: \varphi \in p\}$, $q = \{Q_1(x)\} \cup \{x \neq c_n: n < \omega\}$.

We add to T_1 an axiom of extensionality

$$\varphi_1 = (\forall xy)[(\forall z)[z \in x \leftrightarrow z \in y] \rightarrow x = y]$$