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UNITARY DILATIONS FOR COMMUTING CONTRACTIONS

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Let $S_1, S_2, \cdots, S_n$ be a set of commuting contraction operators on a Hilbert space H, let $U_1, U_2, \cdots, U_n$ be a set of commuting unitary operators on a Hilbert space K containing H, and let P be the projection from K to H. The set $U_1, \cdots, U_n$ is called a set of commuting unitary dilations for $S_1, \cdots, S_n$ provided that

$$PU_1^{m_1}U_2^{m_2} \cdots U_n^{m_n}x = S_1^{m_1}S_2^{m_2} \cdots S_n^{m_n}x$$

for all x in H and for all nonnegative integers $m_1, m_2, \cdots, m_n$. Sz.-Nagy proved that a single contraction has a unitary dilation, and Ando showed that any two commuting contractions possess a pair of commuting unitary dilations. This note presents several counterexamples which disprove the corresponding conjecture for three or more contractions.

In § 3, three commuting contractions, R, S, T are constructed which do not have commuting unitary dilations. The operators R and S each have norm one, while the operator T may be chosen to have any norm between zero and one. However, the proof yielding the counterexample fails completely if the operators R, S, T are replaced by $\lambda R, \lambda S, T$ with $0 < \lambda < 1$, and this raises another question.

It is known that a finite or infinite set of commuting contractions $S_1, S_2, \cdots$ which satisfies $\sum \|S_k\|^2 \leqq 1$ possesses a set of commuting unitary dilations. Thus it appears that the "size" of a set of contractions may be relevant to the existence of commuting unitary dilations; and since two of the contractions in § 3 have norm one, it is conceivable that this example might be only a peculiar "boundary" phenomenon. In § 4 this notion is dispelled by a more complicated example showing that three commuting contractions, each of norm strictly less than one, can fail to have commuting unitary dilations. Although the example of § 4 is in most (but not all) respects more powerful than that of § 3, the latter is presented separately because of its simplicity.

Section 3 also observes that a recent result of Sz.-Nagy and Foias is equivalent to Ando's theorem. Section 5 shows that the counterexamples constructed in this paper to the unitary dilation conjecture cannot be used as counterexamples to another well-known conjecture concerning spectral sets.

2. Notation and preliminaries.

If H is a subspace of a Hilbert space K, the orthogonal projection from K to H will be written as