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PROJECTING ONTO CYCLES IN SMOOTH, REFLEXIVE BANACH SPACES

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This paper deals with operator algebras generated by certain classes of norm 1 projections on smooth, reflexive Banach spaces. For a strictly increasing continuous function $\mathscr{F}$ on the nonnegative reals, the set of "$\mathscr{F}$-projections" gives rise to operator algebras equal to their second commutants. The principal result is that the closed subspace generated by the set of elements Ex, where x is fixed and E runs through a Boolean algebra of $\mathscr{F}$-projections, is the range of a norm 1 projection that commutes with each projection in the Boolean algebra. Sufficient conditions using Clarkson type norm inequalities are given for the commutativity of the set of all $\mathscr{F}$-projections. Examples in Orlicz spaces are given.

1. Projections in smooth spaces. A *normer* of a nonzero element x in a Banach space X is a functional x^* in the dual X^* such that $||x^*|| = 1$ and $||x|| = x^*(x)$. A normer for x always exists; we say that X is *smooth* if every nonzero x has but one normer, denoted $N(x)$. We make the definition $N(0) = 0$.

Proof of the following three lemmas is left to the reader; see, for instance, [5; p. 447].

LEMMA 1. *In a smooth space X, the norming map $N: X \to S^* \cup \{0\}$ has the following properties, where S^* is the unit sphere of X^*.*

(1) *$N(x)$ is the only element of S^* such that $N(x)(x) = ||x||$ if $x \neq 0$.*

(2) *$N(\lambda x) = (|\lambda|/\lambda)N(x)$ for all scalars $\lambda \neq 0$; in particular, $N(\lambda x) = N(x)$ for $\lambda > 0$.*

(3) *In the real case, $N(x)(y) = \lim\,(\lambda \to 0)(||x + \lambda y|| - ||x||)/\lambda$ for $x, y \in X$ and $x \neq 0$.*

LEMMA 2. *If X is a smooth complex Banach space,* $\operatorname{Re} X$ *is also smooth; indeed, for each $x \neq 0$,* $\operatorname{Re} N(x)$ *is the normer of x in* $(\operatorname{Re} X)^*$.

A vector x is said to be *James-orthogonal* to y if $||x + \lambda y|| \geqq ||x||$ for all real numbers λ.

LEMMA 3. *If X is a smooth space, then $N(x)(y) = 0$ if and only if x is James-orthogonal to y in the real case and James-orthogonal to both y and iy in the complex case. If Y is a subspace, then $N(x)(y) = 0 (y \in Y)$ if and only if $||x + y|| \geqq ||x|| (y \in Y)$.*