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ASCOLI'S THEOREM FOR SPACES OF MULTIFUNCTIONS

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The purpose of this paper is to prove that if X and Y are two arbitrary topological spaces and if $M(X, Y; c)$ denotes the space of all multi-valued functions on X to Y with the compact-open topology, then a closed set $\mathscr{F} \subset M(X, Y; c)$ is compact if at each point $x \in X, \mathscr{F}(x) = \cup \{F(x) \mid F \in \mathscr{F}\}$ has a compact closure in Y, and $\mathscr{F}$ is evenly continuous.

The classical theorem of Ascoli [1] asserts that a uniformly bounded, equicontinuous family of functions has a compact closure in the space of continuous functions with the topology of uniform convergence. This theorem has been the center of many papers, notably the works of Gale [2], Myers [5], Weston [6], and Morse-Kelley [3, pp. 233-237]. The purpose of this paper is to establish a similar version of the Ascoli theorem in the space of multi-valued functions.

Let X and Y be two nonvoid topological spaces. Then $F \subset X \times Y$ is said to be a *multifunction* on X to Y, denoted by $F: X \rightarrow Y$, if and only if for each $x \in X$,

1.1. $$\pi_2((\{x\} \times Y) \cap F) \neq \square ,$$

where π_2 is the second projection of $X \times Y$ onto Y, and $\square$ denotes the empty set. We shall write $F(x)$ for the set defined in 1.1. Thus, loosely speaking, a multifunction is a point-to-set correspondence. If A and B are subsets of X and Y respectively, then we denote $F(A) = \cup \{F(x) \mid x \in A\}$ and $F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \square\}$.

DEFINITION 1.2. A multifunction $F: X \rightarrow Y$ is *continuous* if and only if for each open set V in Y, the set $F^{-1}(V)$ is open and the set $F^{-1}(Y - V)$ is closed in X.

Let Y^X be the set of all (single-valued) functions on X to Y, and let $M(X, Y)$ be the set of all multifunctions on X to Y. Note that Y^X is a subset of $M(X, Y)$. For our later convenience in expressions, let us agree that for any $A \subset X$ and $B \subset Y$:

$$(A, B) = \{F \in M(X, Y) \mid F(A) \subset B\} ,$$
$$)A, B(= \{F \in M(X, Y) \mid A \subset F^{-1}(B)\} .$$

Recall that the compact-open topology [3, p. 221] for Y^X has as a subbase the totality of sets $(K, U) \cap Y^X$ where K is a compact subset