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PERMUTATIONS, MATRICES, AND GENERALIZED YOUNG TABLEAUX

DONALD E. KNUTH

A generalized Young tableau of "shape" $(p_1, p_2, \cdots, p_m)$, where $p_1 \geqq p_2 \geqq \cdots \geqq p_m \geqq 1$, is an array Y of positive integers y_{ij}, for $1 \leqq j \leqq p_i, 1 \leqq i \leqq m$, having monotonically nondecreasing rows and strictly increasing columns. By extending a construction due to Robinson and Schensted, it is possible to obtain a one-to-one correspondence between $m \times n$ matrices A of nonnegative integers and ordered pairs (P, Q) of generalized Young tableaux, where P and Q have the same shape, the integer i occurs exactly $a_{i1} + \cdots + a_{in}$ times in Q, and the integer j occurs exactly $a_{1j} + \cdots + a_{mj}$ times in P. A similar correspondence can be given for the case that A is a matrix of zeros and ones, and the shape of Q is the transpose of the shape of P.

Figure 1 shows two arrangements of integers which we will call *generalized Young tableaux of* shape (6, 4, 4, 1). A *generalized Young tableau of shape* $(p_1, p_2, \cdots, p_m)$ is an array of $p_1 + p_2 + \cdots p_m$ positive integers into m left-justified rows, with p_i elements in row i, where $p_1 \geqq p_2 \geqq \cdots \geqq p_m$; the numbers in each row are in nondecreasing order from left to right, and the numbers in each column are in strictly increasing order from top to bottom. (The special case where the elements are the integers $1, 2, \cdots, N = p_1 + p_2 + \cdots + p_m$, each used exactly once, was introduced by Alfred Young in 1900 as an aid in the study of irreducible representations of the symmetric group on N letters; see [6].)

Consider on the other hand the 6×7 array

$$(1.1) \qquad A = \begin{pmatrix} 0&0&1&0&0&0&0 \\ 0&0&0&0&0&2&0 \\ 1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0 \\ 2&1&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}$$

having respective column sums $(c_1, \cdots, c_7) = (3, 2, 3, 2, 1, 3, 1)$ and row sums $(r_1, \cdots, r_6) = (1, 2, 5, 2, 4, 1)$. Note that in Figure 1 the integer i occurs r_i times in Q, and the integer j occurs c_j times in P. In this paper we shall give a constructive procedure which yields a one-to-one correspondence between matrices A of nonnegative integers