PACIFIC JOURNAL OF MATHEMATICS
Vol. 35, No. 1, 1970

A CHARACTERIZATION OF THE NIL RADICAL OF A RING

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Let R be a ring and S a subring of R. Let φ be a ring homomorphism mapping S onto a division ring Γ. Choose an ideal $P \subsetneqq R$ maximal with respect to the property $(P \cap S)^{\phi} = (0)$. P is a prime ideal of R. If P is any prime ideal of R which can be obtained in the above manner write $P = P(\Gamma, S, \varphi)$.

It is shown that all primitive ideals are of the form $P = P(\Gamma, S, \varphi)$ and that a ring R is nil if and only if it has no prime ideals of the form $P = P(\Gamma, S, \varphi)$. It is shown that the nil radical of any ring is the intersection of all prime ideals $P = P(\Gamma, S, \varphi)$.

It is shown that if $P = P(\Gamma, S, \varphi)$ for all prime ideals $P \subsetneqq R$ then the nil and Baer radicals coincide for all homomorphic images of R. If the nil and Baer radicals coincide for all homomorphic images of R, it is shown that any prime ideal P of R is contained in a prime ideal $P' = P'(\Gamma, S, \varphi)$.

Finally, by consideration of prime ideals $P = P(\Gamma, S, \varphi)$, two theorems are proved giving information about rings satisfying very special conditions.

2. Certain prime ideals in rings. Let R be any ring and S a subring of R. Suppose φ is a ring homomorphism mapping S onto a division ring Γ. We may choose an ideal $P \subsetneqq R$ maximal with respect to the property $(P \cap S)^{\varphi} = (0)$. It is an easy exercise to check that P will be a prime ideal of R. If P is any prime ideal of R which is a maximal ideal such that $(P \cap S)^{\varphi} = (0)$ for some subring $S \subsetneqq R$ and some ring homomorphism $\varphi: S \rightarrow \Gamma$, Γ a division ring, we write $P = P(\Gamma, S, \varphi)$. Throughout, for any ring R, we let $J(R)$, $N(R)$, $\beta(R)$ denote respectively the Jacobson, nil, and Baer radicals of R. We start with the following simple fact.

THEOREM 1. *Let R be a ring and P a primitive ideal of R. Then $P = P(\Gamma, S, \varphi)$.*

Proof. Let $P = (0: M)$ for some simple right R module M. Let Γ be the centralizer of M. Γ is a division ring. As R/P is primitive it is well known ([3], Th. 3, p. 33) that there exists a subring $S' \subsetneqq R/P$ and a homomorphism $\varphi': S' \rightarrow \Gamma$. It is easy to check $P = P(\Gamma, S, \varphi)$ with $S = (S')\pi^{-1}$, $\varphi = \pi\varphi'$, π the natural map from R onto R/P.