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TORSION-FREE AND DIVISIBLE MODULES OVER MATRIX RINGS

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A short exact sequence $0 \to K \to F \to E \to 0$ of left modules over a ring A is 1-pure if $aK = K \cap aF$ for all $a \in A$, and *pure* if for any right A-module M, the map $M \otimes K \to M \otimes F$ is injective. A module E is torsion free (Hattori) if its presence on the right forces 1-purity, and flat if it forces purity. Similarly, we have on the left the notions of divisibility (Hattori) and absolute purity. Considering the functor $E \to E^n$ taking A-modules to modules over the matrix ring $M_n(A)$, a sequence is called *n-pure* if its image under this functor is 1-pure; *n-torsion-free* and *n-divisible* modules are similarly defined. It is shown that purity, flatness, and absolute purity, respectively, are equivalent to the requirement that n-purity, n-torsion-freeness, and n-divisibility should hold for all n. n-divisibility and absolute purity are preserved under direct sums, products and certain inductive limits; n-torsion-freeness and flatness under direct sums and inductive limits, but not products. A condition is given guaranteeing that products of at most a given cardinality preserve n-torsion-freeness. It is shown that if every left ideal of A is generated by at most n elements, then n-torsion-freeness is equivalent to flatness. The behavior of these properties under localization is studied, and it is shown that if A is locally a domain then the two notions of purity agree if and only if w. gl. dim. $(A) \leqq 1$.

A will always denote a ring with identity; all modules will be unitary and left modules unless otherwise stated. If no confusion can arise $F \otimes E$ will mean $F \otimes_A E$; similarly for Hom (F, E), $\mathrm{Tor}_m(F, E)$, and $\mathrm{Ext}^m(F, E)$.

1. Matrices. For a positive integer n, let $M_n(A)$ denote the ring of $n \times n$ matrices over A (we shall sometimes use $B = M_n(A)$ for convenience of notation) and $M_n(E)$ the left $M_n(A)$-module of $n \times n$ matrices over E, where scalar multiplication looks like usual matrix multiplication. Let $e_{ij} \in M_n(A)$ be the matrix having 1 in the (i, j) position and zeros elsewhere.

When considering E^n as a left $M_n(A)$-module, it is convenient to think of the elements as "column vectors", so we will denote an n-tuple of E^n as $(x_1, x_2, \cdots, x_n)'$, the prime denoting transpose. Note that $M_n(E)$ is a direct sum (as $M_n(A)$-modules) of n copies of E^n.