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CARATHEODORY THEOREMS IN CONVEX PRODUCT STRUCTURES

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Various attempts have been made to place convexity in an axiomatic setting. Recently J. Eckhoff has considered the classic theorem of Radon in several different settings. Most of his work is done in what we call an Eckhoff space, i.e., in a finite product of euclidean spaces where convex sets are defined as the cartesian products of usual convex sets in each component space. The purpose of this paper is to investigate the closely related theorem of Caratheodory and its generalizations in this setting.

The papers of F. W. Levi [5] and Dauzer, Grunbaum, Klee [3] have various approaches to axiomatic settings of convexity, and a good bibliography for before 1961. See the papers of Eckhoff [4] and Bonnice-Reay [2] for more recent results and references.

1. Eckhoff spaces. The pair $(E, \mathscr{C})$ denotes an *Eckhoff space* provided (1) E is a direct cartesian product $E = \prod_{i=1}^{n} E_i$ where each E_i is a d_i-dimensional euclidean space with $\mathscr{C}_i$ the family of all convex sets of E_i, and (2) $\mathscr{C} = \{\prod_{i=1}^{n} A_i : A_i \in \mathscr{C}_i\}$ is the family of all *product-convex* sets in E. For any set $X \subset E$, the set $E(X) = \cap\{A : X \subset A \in \mathscr{C}\}$ is called the *product-convex hull* of X. Let $\pi_i : E \to E_i$ denote the usual projection. Then we can consider E as a linear space of dimension $d = \sum_{i=1}^{n} d_i$, and $E(X) = \prod_{i=1}^{n} (\operatorname{conv} \pi_i X)$ where $\operatorname{conv} B$ denotes the usual convex hull of B in each euclidean space E_i. The cardinality of B will be denoted by $|B|$. Using the notation of Bonnice-Klee [1] and others, we say that $\operatorname{int}_r B$ is the set of all points p for which there exists an r-dimensional simplex contained in B and containing p in its relative interior.

2. Caratheodory-type theorems. By a Caratheodory-type theorem we mean a result which asserts that if a point is embedded in the (axiomatically defined) hull of a set X, then it is similarly embedded in the hull of a sufficiently small subset of X. Note that the case $n = 1$ of Theorem 1 below is the result usually called Caratheodory's theorem.

THEOREM 1. *If X is any subset of an Eckhoff space $E = \prod_{i=1}^{n} E_i$ of dimension $d = \Sigma d_i$ and if $p \in E(X)$, then $p \in E(Y)$ for some $Y \subset X$ with $|Y| \leq d + \delta$, where $\delta = 1$ if $n = 1$ and $\delta = 0$ if $n > 1$. Fur-*