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ON MATRICES WITH A RESTRICTED NUMBER OF DIAGONAL VALUES

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This note confirms the following conjecture of Marcus: Let $A = (a_{ij})$ be an $n \times n$ matrix of strictly positive entries with at most $(n-1)$ distinct diagonal values, then A is singular. We also show that there exist matrices with strictly positive entries with n diagonal values which are nonsingular.

DEFINITIONS. If A is an $n \times n$ matrix and σ is a permutation of $\{1, 2, \cdots, n\}$, then the product $a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$ is called the *diagonal* of A corresponding to σ.

If A_1, A_2 are two $n \times n$ matrices, then A_1 is called a *diagonate* of A_2 if A_1 can be obtained from A_2 by a finite number of operations of the following kinds:

(i) Multiplication of all entries of some row, (or column) by some $c > 0$.

(ii) Interchange of any two rows (or columns).

The notation $A[\mu \mid \gamma]$, $A(\mu \mid \gamma)$ is that of [1].

PRELIMINARY REMARKS. (i) The property of being a diagonate is an equivalence relation.

(ii) If a matrix is singular (nonsingular), then each of its diagonates is singular (nonsingular).

(iii) If a matrix A_1 has diagonal values $\rho_1 < \rho_2 < \cdots < \rho_r$ then a diagonate A_2 of A_1 has diagonal values $k\rho_1 < k\rho_2 < \cdots k\rho_r$, where $k = k(A_2)$, and $|\det A_1| = |k \det A_2|$.

(iv) If a matrix has strictly positive (positive) entries, then each of its diagonates has strictly positive (positive) entries.

LEMMA. *If $X = (x^{e(i,j)})$ is an $n \times n$ matrix with entries in an extension $F(x)$ of the real field F, where $e(i, j)$ are nonnegative rational integers $i, j = 1, 2, \cdots n$ and $e(1, j) = 0$ for $j = 1, 2, \cdots n$, then*

$\det X = (x-1)^{n-1} g(x)$, *where $g(x)$ is a polynomial in x with rational integral coefficients.*

The proof of the lemma is by induction. The result is trivial for $n = 2$. The result is therefore assumed to hold for all $n < N$, and $N > 2$. If $n = N$, subtracting the first row of X from the second and expanding X by its second row, we have