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ON THE NUMBER OF FINITELY GENERATED 0-GROUPS

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Let K be a class of relational systems of a fixed similarity type, $\mathfrak{n}$ an infinite cardinal. A system $\mathfrak{A}$ of cardinality $\mathfrak{n}$ is $(\mathfrak{n}, K)$-weakly universal if each system in K of cardinality at most $\mathfrak{n}$ is isomorphically embeddable in A. The object of this note is to construct $2^{\aleph_0}$ nonisomorphic finitely generated 0-groups and hence answer in the negative the following problem attributed to B. H. Neumann. Is there a group which is $(\aleph_0, K_1)$-weakly universal, where K_1 is the class of 0-groups?

If $\mathfrak{A}$ is $(\mathfrak{n}, K)$-weakly universal and also a member of K, then $\mathfrak{A}$ is $(\mathfrak{n}, K)$-universal. It is known that $(\mathfrak{n}, K)$-universal systems exist for many classes K and cardinals $\mathfrak{n}$. In particular, Morley and Vaught established a useful condition for the existence of $(\mathfrak{n}, K)$-universal systems for K an elementary class, $\mathfrak{n}$ an appropriate cardinal (see [7]). However there are no theorems of wide applicability concerning the existence of $(\aleph_0, K)$-universal systems; here the structure of the systems in K must be carefully analyzed. To illustrate this, consider the classes K_1 of 0-groups; K_2 of abelian 0-groups (i.e., torsion free abelian groups); K_3 of ordered groups (i.e., groups of type $\langle H, \cdot, \leqq \rangle$ where $\langle H, \cdot \rangle$ is an 0-group linearly ordered by $\leqq$); K_4 of abelian ordered groups. By applying the results in [7], (assuming the generalized continuum hypothesis), it is easily seen that there exists an $(\mathfrak{n}, K_i)$-universal system for all $\mathfrak{n} > \aleph_0$ and $i = 1, 2, 3$, or 4.

The situation for $\mathfrak{n} = \aleph_0$ is more complicated. There is an $(\aleph_0, K_2)$-universal group (see [1, p. 64]). However, there is no ordered group which is $(\aleph_0, K_4)$-weakly universal and hence there is no $(\aleph_0, K_3)$-universal group. This follows readily from the fact that the free abelian group on two generators has $2^{\aleph_0}$ nonisomorphic orders (see [2, p. 50]). Theorem 2, which establishes the nonexistence of a group which is $(\aleph_0, K_1)$-weakly universal, solves a problem of B. H. Neumann (see [2, p. 211, Problem 17]).

1. Definitions. An 0-group is a group G for which there exists a linear ordering relation $\leqq$ on G satisfying the following condition: $a \leqq b$ implies $c\,a\,d \leqq c\,b\,d$ for all $a, b, c, d \in G$. For a group G the commutator of x and y in G is denoted $[x, y] = x^{-1}\,y^{-1}\,xy$; for subsets A and B of G, $[A, B]$ is the subgroup of G generated by $\{[a, b] : a \in A, b \in B\}$; $G' = [G, G]$; $G'' = [G', G']$. Let F be the free