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ALMOST SMOOTH PERTURBATIONS OF SELF-ADJOINT OPERATORS

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Assume $H^0 \in \mathscr{C}(\mathfrak{H})$ is a self-adjoint operator with spectrum on $[o, \infty)$ and that $E^0(\Delta) \in \mathscr{B}(\mathfrak{H})$ is the spectral measure determined by $H^0, \Delta \subset [0, \infty)$. Let $H^1 = H^0 + V$ where $V = B \cdot A$ and $A, B \in \mathscr{B}(\mathfrak{H})$ are commuting self-adjoint operators. In this paper T. Kato's concept of smooth perturbations is generalized in the following way: H^1 is said to be an almost smooth perturbation of H^0, except at $1 = 0$, if A, B are smooth with respect to $H^0E^0(\Delta_m)$ for all intervals $\Delta_m = (1/m, \infty), m \geqq 1$. It is proved that the time independent wave operators corresponding to H^0, H^1 exist when the assumption that H^1 is smooth with respect to H^0 is replaced by the assumption that H^1 is almost smooth with respect to H^0.

The concept of smooth perturbations was introduced by T. Kato in [2]. The importance of the generalization given here is that it allows one to apply the theory developed in [2] to certain one dimensional differential operators which are almost smooth but not smooth. Examples of some almost smooth ordinary differential operators are given below in § 3.

2. **The wave operators.** Let $\Omega_\pm$ denote the upper and lower complex plane, with the reals excluded, and let f be a function on $\Omega_\pm \times \mathfrak{H}$ into $\mathfrak{H}$. Such a function f is said to be in the Hardy class $H_2((-\infty, \infty): \mathfrak{H})$ if and only if f is analytic in λ for all $\lambda \in \Omega_\pm$ and for all $u \in \mathfrak{H}, \delta > 0$, $\int_{-\infty}^{\infty} ||f(1 \pm i\delta; u)||^2 d1 \leqq P||u||^2$ for some $P > 0$ independent of u and δ. An operator $A \in \mathscr{B}(\mathfrak{H})$ is said to be smooth with respect to H^0 if and only if the function f defined by

$$(\lambda, u) \mapsto A(H^0 - \lambda I)^{-1}u = AR^0(\lambda)u \text{ is in } H_2((-\infty, \infty): \mathfrak{H})$$

[2, p. 260].[1] Now we shall make the following assumptions regarding H^0, A, B:

(i) For some N, $||BR^0(\lambda)E^0(\Delta_m)A|| \leqq K < 1$ for $m \geqq N$ and for all λ not real positive or zero.

(ii) $H^1 = H^0 + BA$ is an almost smooth perturbation of H^0. It will be shown below that these two assumptions insure the existence of the wave operators in the time independent form. With additional assumptions, one may also show that these operators coincide with

[1] Actually T. Kato defines smoothness for more general operators than those considered in this paper.