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ON COMMUTATIVE ENDOMORPHISM RINGS

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This note deals with a finitely generated faithful module E over a commutative semi-prime noetherian ring R, with commutative endomorphism ring $\mathrm{Hom}_R(E, E) = \Omega(E)$. It is shown that E is identifiable to an ideal of R whenever $\Omega(E)$ lacks nilpotent elements; a class of examples with $\Omega(E)$ commutative but not semi-prime is discussed.

1. Main result. Throughout R will denote a commutative noetherian ring and modules will be finitely generated. In order to use the full measure of the ring, we shall consider mostly faithful modules. As for notation, unadorned $\otimes$ and Hom are taken over the base ring.

In case R is semi-prime (meaning here: no nilpotent elements distinct from 0) we recall that its total ring of quotients K is semi-simple, and thus a direct sum of fields $K = \oplus\sum K_i$, $1 \leqq i \leqq n$. Any ideal I of R has the property that $\mathrm{Hom}(I, I)$ is commutative and semi-prime: for if S denotes the set of regular elements of R,

$$\mathrm{Hom}(I, I) \subseteqq \mathrm{Hom}(I, I)_S = \mathrm{Hom}_{R_S}(I_S, I_S) .$$

But this last is a subring of K. The content of the next theorem is precisely a converse to this observation.

THEOREM 1.1. *Let E be a finitely generated faithful module over the semi-prime ring R. Then, if* Hom (E, E) *is commutative and semi-prime, E is isomorphic to an ideal of R.*

Proof. Denote by T the torsion submodule of E, i.e., let T be the set of elements of E annihilated by a regular element of R. If $T = 0$, then $\mathrm{Hom}(E, E) \subseteqq \mathrm{Hom}(E, E)_S = \mathrm{Hom}_{R_S}(E_S, E_S)$; using the decomposition of $R_S = K$ as a direct sum of fields,

$$\mathrm{Hom}_K(E \otimes K, E \otimes K) = \oplus\sum \mathrm{Hom}_{K_i}(E \otimes K_i, E \otimes K_i) .$$

Since $\mathrm{Hom}_K(E \otimes K, E \otimes K)$ is commutative, we must have, for each i, $E \otimes K_i = 0$ or isomorphic to K_i. This allows identification of E_S to a submodule of K and consequently of E to an ideal of R, since E is finitely generated.

Assume then, by way of contradiction, $T \neq 0$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \xrightarrow{\pi} F \longrightarrow 0 .$$