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THE CONVEX GENERATION OF CONVEX BOREL SETS IN EUCLIDEAN SPACE

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The main object of this note is to resolve a problem of V. L. Klee on the convex generation of convex Borel sets in Euclidean space, into a sequence of three problems, each of some intrinsic interest; the joint solution of the three problems being equivalent to the solution of Klee's problem.

In general terms, Klee asks the question whether a convex Borel set can be generated by the Borel method from the closed (or open) convex sets without at any stage leaving the domain of convex sets. More specifically he calls a set K a convexly generated Borel set if:

A. K belongs to the minimal system of sets, containing the closed convex sets, that is closed under the operations of countable increasing union and of countable decreasing intersection.

Naturally the operations of countable increasing union and of countable decreasing intersection lead from the class of convex sets to the class of convex sets. Klee asks whether the class of convexly generated Borel sets coincides with the class of convex sets that are Borel sets. D. G. Larman [6] has recently given the answer 'yes' in 3-dimensional space, using methods that do not readily generalize.

It is not quite clear why Klee works with condition A. It might seem equally reasonable to work with the sets K satisfying the condition that:

B. K belongs to the minimal system of sets, containing the closed convex sets, that is closed under the operations of countable increasing union and of countable intersection.

Here of course it is the penultimate word of A that has been omitted. D. G. Larman [7] has shown that these two conditions are equivalent.

But there are other operations that lead from convex Borel sets to convex Borel sets. We show, in § 2 below, that the smallest convex set containing two convex Borel sets is a convex Borel set, by a method that seems to depend essentially on the properties of Euclidean space. This result is perhaps a little surprising in view of the facts that the smallest convex set containing a Borel set is not necessarily a Borel set (even in E_3) and that the vector sum of two Borel sets on the line is not necessarily a Borel set (see [8] and [1]).

Thus it is reasonable to call a set K a weakly convexly generated Borel set if it satisfies the condition: