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MACDONALD'S THEOREM FOR QUADRATIC JORDAN ALGEBRAS

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Macdonald's Theorem says that if an identity in three variables x, y, z which is linear in z holds for all special Jordan algebras, it holds for all Jordan algebras. We show this is equivalent to saying the universal quadratic envelope $\mathscr{U}\mathscr{Q}\mathscr{E}(\mathfrak{J}^{(2)})$ of the free Jordan algebra $\mathfrak{J}^{(2)}$ on two generators x, y is canonically isomorphic to the universal compound linear envelope $\mathscr{U}\mathscr{C}\mathscr{E}(\mathfrak{J}^{(2)})$. We generalize Macdonald's Theorem from the case of linear Jordan algebras over a field of characteristic $\neq 2$ to quadratic Jordan algebras over an arbitrary ring of scalars, at the same time improving on the results in the linear case by presenting $\mathscr{U}\mathscr{Q}\mathscr{E}(\mathfrak{J}^{(2)})$ in terms of a finite number of generators and relations. Similarly we generalize Macdonald's Theorem with Inverses concerning identities in x, x^{-1}, y, y^{-1}, z. Finally, we prove Shirshov's Theorem that $\mathfrak{J}^{(2)}$ is special.

PART I. MACDONALD'S THEOREM.

1. Free algebras and free products. Throughout this paper Φ will denote a fixed ring of scalars (=unital commutative, associative ring), and "linear space", "linear map", etc. will always mean linear with respect to Φ.

Recall [4, p. 000] that a *unital quadratic algebra* $\mathfrak{Q} = (\mathfrak{X}, U, 1)$ is a linear space $\mathfrak{X}$ together with a quadratic mapping $x \to U(x) = U_x$ of $\mathfrak{X}$ into $\mathrm{Hom}_\Phi(\mathfrak{X}, \mathfrak{X})$ and a *unit element* $1 \in \mathfrak{X}$ satisfying $U_1 x = x$ and $\{x\,1\,y\} = \{x\;y\;1\}$ for all x, y (where, as usual, $\{x\;y\;z\} = U_{x,z}y = \{U_{x+z} - U_x - U_z\}y$ is trilinear). A *homomorphism* $\varphi: \mathfrak{Q} \to \tilde{\mathfrak{Q}}$ is a linear map satisfying

$$\varphi(1) = \tilde{1} \quad \varphi(U_x y) = \tilde{U}_{\varphi(x)}\varphi(y) .$$

An *ideal* is a subspace $\mathfrak{R} \subset \mathfrak{Q}$ such that $U_{\mathfrak{R}}\mathfrak{Q} \subset \mathfrak{R}$, $U_{\mathfrak{Q}}\mathfrak{R} \subset \mathfrak{R}$, $\{\mathfrak{R}\mathfrak{Q}\mathfrak{Q}\} \subset \mathfrak{R}$.

Given any set X we can construct a *free unital quadratic algebra* $\mathscr{F}\mathscr{Q}(X)$ on X with an imbedding $i: X \to \mathscr{F}\mathscr{Q}(X)$ having the following universal property: any (set-theoretic) map $\varphi: X \to \mathfrak{Q}$ of X into a unital quadratic algebra $\mathfrak{Q}$ extends uniquely to a homomorphism $\hat{\varphi}: \mathscr{F}\mathscr{Q}(X) \to \mathfrak{Q}$, i.e., $\varphi = \hat{\varphi} \circ i$. The construction goes as follows [1, p. 116]. We recursively define monomials in the elements of X, starting with the empty monomial 1 of degree 0 and the monomials $x \in X$ of degree 1, and using monomials m, n, p of degrees i, j, k to form new monomials $(m; n)$ of degree $2i + j$ and $(m, n; p) = (n, m; p)$ of degree $i + j + k$; we identify $(1; m)$ with m, $(m, n; 1)$ with $(m, 1; n)$,