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THE SUSLIN-KLEENE THEOREM FOR V_κ WITH COFINALITY(κ) = ω

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It is easy to extend to arbitrary structures $\mathfrak{A} = \langle A, R_1, \cdots, R_l f_1, \cdots, f_m \rangle$ the concepts of $\prod_1^1$ and inductively definable relations, which are familiar for the structure of the integers. The second author showed in a recent paper that these two concepts coincide for countable $\mathfrak{A}$ that satisfy certain mild definability conditions—this is a generalization of the classical Suslin-Kleene theorem. Here we generalize the Suslin-Kleene theorem in a different direction.

MAIN RESULT. Let V_κ be the set of sets of rank less than κ, i.e., $V_0 = \phi$, $V_{\xi+1} =$ power of V_ξ, $V_\kappa = \bigcup_{\xi<\kappa} V_\xi$, if κ is limit. The classes of inductively definable and $\prod_1^1$ relations on the structure $\mathscr{V}_\kappa = \langle V_\kappa, \in \restriction V_\kappa \rangle$ $(\kappa \geq \omega)$ coincide if and only if κ is a limit ordinal with cofinality ω.

This implies several corollaries about the class of $\prod_1^1$ relations on V_κ, when cofinality$(\kappa) = \omega$, e.g., that it has the reduction property.

The nontrivial part of the theorem is the implication $\prod_1^1 \Rightarrow$ *inductively definable* for $\mathscr{V}_\kappa$ with *cofinality*$(\kappa) = \omega$.

1. Proof of the main result.

We assume familiarity with [7], whose notation we shall use.

Notice first that for each $\kappa \geq \omega$, $\mathscr{V}_\kappa$ is an *acceptable* structure, in the sense of [7]. This is immediate for limit κ, by taking the ordinary set-theoretic pair and the standard ω for the integers within V_κ. For successor κ the proof is by induction; let $\kappa = \lambda + 1$, let $(\ ,\ )_\lambda$ be a definable pair in $\mathscr{V}_\lambda$, for $x_1, \cdots, x_n$ in V_κ put

$$\langle x_1, \cdots, x_n \rangle = \{(1, u)_\lambda : u \in x_1\} \cup \cdots \cup \{(n, u)_\lambda : u \in x_n\} .$$

These n-tuple functions are definable in $\mathscr{V}_\kappa$ and using them one can easily define a pair for $\mathscr{V}_\kappa$ and also show that first-order definability on $\mathscr{V}_\kappa$ is preserved under inductive definitions.

Since $\mathscr{V}_\kappa$ is acceptable, the inductively definable relations on $\mathscr{V}_\kappa$ are $\prod_1^1$ by the argument given in §3 of [7]. Also, if $\kappa \geq \omega$ and κ is a successor or cofinality$(\kappa) > \omega$, then the relation

$$S \in WF \Leftrightarrow \textit{there is no sequence } u_0, u_1, \cdots, \textit{so that } (n)[(u_n, u_{n+1}) \in S]$$

is first-order definable on $\mathscr{V}_\kappa$, so that by the usual analysis of trans-