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AN ELEMENTARY PROOF OF THE UNIQUENESS OF THE FIXED POINT INDEX

ROBERT F. BROWN

In 1953, Barrett O'Neill stated axioms for a "fixed point index" and obtained existence and uniqueness theorems for the index on finite polyhedra. The proof of uniqueness consisted of showing that any function which satisfied the axioms must agree with the index he had already defined. This paper presents a proof of the uniqueness of the fixed point index on finite polyhedra which depends only on the axioms and therefore is "elementary" in the sense that it is independent of the existence of an index. The proof is "elementary" also in that all the techniques used are taken from geometric topology or calculus so that, in particular, no algebraic topology is required. An elementary proof of the uniqueness of the fixed point index on compact metric absolute neighborhood retracts is an immediate consequence of the material in this paper.

1. **The axioms.** Let $\mathscr{C}$ be a collection of topological spaces. Denote by $\mathscr{C}'$ the collection of all triples (X, f, U) where X is in $\mathscr{C}$, $f: X \to X$ is a map, and U is an open subset of X such that there are no fixed points of f on the boundary of U.

A *fixed point index on* $\mathscr{C}$ is a function $i: \mathscr{C}' \to Z$ (the integers) such that

I. (*Localization*). If $(X, f, U) \in \mathscr{C}'$ and $g: X \to X$ is a map such that $g(x) = f(x)$ for all x in the closure of U, then $i(X, f, U) = i(X, g, U)$.

II. (*Homotopy*). Given a map $H: X \times I \to X$, define $h_t: X \to X$, for $t \in I = [0, 1]$, by $f_t(x) = H(x, t)$. If $(X, h_t, U) \in \mathscr{C}'$ for all $t \in I$, then $i(X, h_0, U) = i(X, h_1, U)$.

III. (*Additivity*). If $(X, f, U) \in \mathscr{C}'$ and $U_1, \cdots, U_s$ is a set of mutually disjoint open subsets of U such that $f(x) \neq x$ for all $x \in [U - \bigcup_{j=1}^{s} U_j]$, then $i(X, f, U) = \sum_{j=1}^{s} i(X, f, U_j)$.

IV. (*Weak Normalization*). If $(X, f, U) \in \mathscr{C}'$ where f is the constant map such that $f(X) = x_0 \in U$, then $i(X, f, U) = 1$.

V. (*Commutativity*). If X and X' are in $\mathscr{C}$ and $f: X \to X'$, $g: X' \to X$ are maps such that $(X, gf, U) \in \mathscr{C}'$ then $i(X, gf, U) = i(X', fg, g^{-1}(U))$.

The axiom list above is a modified version of the one used by Browder in [1]. It differs somewhat from O'Neill's original list [5].

2. **Proof of uniqueness.** Given a space X and a function $f: X \to R^n$ (euclidean n-dimensional space), we will always denote by