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THE ABEL SUMMABILITY OF CONJUGATE MULTIPLE FOURIER-STIELTJES INTEGRALS

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Let $K(x) = \Omega(x/|x|)|x|^{-k}$ where $\Omega(\xi)$, $|\xi| = 1$, is a real valued function which is in Lip α, $0 < \alpha < 1$, on the unit $(k-1)$-sphere S in k-dimensional Euclidean space, E_k, $k \geqq 2$ with the additional property that $\int_S \Omega(\xi)d\sigma(\xi) = 0$ where σ is the natural surface measure for S. ($K(x)$ is usually called a Calderón-Zygmund kernel in Lip α.) Let μ be a Borel measure of finite total variation on E_k and set $\hat{\mu}(y) = (2\pi)^{-k}\int_{E_k} e^{-i(y,w)}d\mu(w)$. Also designate the principal-valued Fourier transform of K by $\hat{K}(y)$ and the principal-valued convolution of K with μ by $\tilde{\mu}(x)$. Define $I_R(x) = (2\pi)^k\int_{E_k} e^{-|y|/R}\hat{K}(y)\hat{\mu}(y)e^{i(y,x)}dy$. Then if k is an even integer or if $k = 3$, the following result is established: $\lim_{R\to\infty} I_R(x) = \tilde{\mu}(x)$ almost everywhere.

In [5] V. L. Shapiro proved that the conjugate Fourier-Stieltjes integral of a finite Borel measure μ in the plane E_2, taken with respect to a Calderón-Zygmund kernel $K(x)$ in Lip α, $1/2 < \alpha < 1$, is almost everywhere Abel summable to the principal-valued convolution $K*\mu$. The purpose of this paper is to extend this result to E_3 and to even-dimensional E_k for $K(x)$ in Lip α, $0 < \alpha < 1$. The first author will obtain the corresponding result for the odd-dimensional cases $k = 2s + 1$, $s \geqq 2$, in a paper to appear, by the use of special functions. Also, the results of the present paper should be compared with Theorem 2 of [6, p. 44].

2. Definitions and notation. For $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k)$ put $(x, y) = x_1y_1 + \cdots + x_ky_k$, $|x| = (x, x)^{1/2}$ and $B(x, t) = \{y: |x - y| < t\}$. We will work with a fixed Calderón-Zygmund kernel $K(x) = \Omega(x/|x|)/|x|^k$ where $\Omega(\xi)$, $|\xi| = 1$, is a real-valued function defined on the unit $(k-1)$-dimensional sphere S in Euclidean space E_k, $k \geqq 2$, and $\int_S \Omega(\xi)d\sigma(\xi) = 0$, where σ is the natural surface measure for S [2, Chapter 11]. We define $K(x)$ to be in Lip α if $|\Omega(\xi) - \Omega(\eta)| = 0(|\xi - \eta|^\alpha)$ for some α, $0 < \alpha < 1$. The Fourier transform of a Borel measure μ in E_k of finite total variation is denoted as usual by

$$(1) \qquad \hat{\mu}(y) = (2\pi)^{-k}\int_{E_k} e^{-i(y,w)}d\mu(w)$$

and by the principal-valued convolution $\tilde{\mu}(x)$ we mean