PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 1, 1971

CHARACTERS AND ORTHOGONALITY IN FROBENIUS ALGEBRAS

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Matrix-theoretical proofs of orthogonality relations for the coefficients of representations of Frobenius algebras have been extensively developed in the literature. This paper has grown out of a desire to prove some of these orthogonality relations in a matrix-free, module-theoretic manner. Most of the attention is focused on characters, even though some of the results diverge from this restriction.

Sections 1 and 2 set the stage for the development of the principal theorems in § 3. There we derive the orthogonality relations and demonstrate a relation between characters and certain homogeneous modules over Frobenius algebras. In § 4 we apply these results to obtain some information about the character of the left regular module.

All rings are assumed to have an identity, and all modules are assumed to be unital. The Jacobson radical of a ring A will be written $J(A)$, or simply J.

Let A be a ring, M a simple left A-module, and let $A_M = \{a \in A: aM = 0\}$. Then A_M is a two-sided ideal of A, the *annihilator* of M. If L is a left A-module, $\mathrm{Soc}_M(L)$ will denote the *M-socle of L*, *i.e.*, the sum of all submodules of L isomorphic to M. We say L is *M-homogeneous* if $\mathrm{Soc}_M(L) = L$. In particular if A is left artinian, then A_M is a maximal two-sided ideal of A, and L is M-homogeneous if and only if $A_M L = 0$. We let S_M denote the M-socle $\mathrm{Soc}_M(A)$ of A. A *block* of A is an indecomposable ring direct summand of A [2, § 55]. We say *M belongs to* the block B of A if M is a composition factor of B regarded as a left A-module.

Let K be a field. If V is a vector space over K, let $(V:K)$ denote the K-dimension of V. Now assume A is a finite dimensional K-algebra. Then $A^* = \mathrm{Hom}_K(A, K)$ is an (A, A)-bimodule, where for $a \in A$ and $\lambda \in A^*$ we define

$$\begin{cases}(a\lambda)(x) = \lambda(xa) \\ (\lambda a)(x) = \lambda(ax)\ .\end{cases} \qquad (x \in A)$$

If $\lambda \in A^*$, we say λ is a *class function* if $\lambda(ab) = \lambda(ba)$ for all $a, b \in A$. Let $\mathrm{cf}(A)$ denote the set of class functions in A^*. Observe that $\lambda \in \mathrm{cf}(A)$ if and only if $a\lambda = \lambda a$ for all $a \in A$. We say $\chi \in A^*$ is an *A-character* if χ is the character of a (finite dimensional) left A-module. Clearly all A-characters belong to $\mathrm{cf}(A)$. The A-character χ is said to be *irreducible* if it is the character of a simple left A-module.