PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 1, 1971

ANALYTIC AND HARMONIC OBSTRUCTION ON NONORIENTABLE KLEIN SURFACES

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It is well-known that, on a finitely connected, noncompact, Riemann surface, the complex-dimension of the space of all analytic differentials modulo the space of exact analytic differentials is the first Betti number of the underlying surface, and hence its real-dimension twice the first Betti number. Further, it is well-known that the group of units of the algebra of analytic functions on such a surface modulo the subgroup of exponential functions is a free Abelian group whose rank is again the first Betti number of the underlying surface. Thus, in each case, the analytic obstruction on the surface fully dualizes the continuous obstruction.

Interestingly, this is not the case on a finitely connected, noncompact, nonorientable Klein surface; for example, in the case of the first problem, the real-dimension is twice the first Betti number minus one. In the second problem, the group in question is isomorphic to the direct sum of the two element group and the free Abelian group whose rank is the first Betti number, of the underlying space, minus one. These calculations are first made using sheaf theory, in §2. Integration theory is then applied, §3, to elucidate the reason that this curious defect occurs. Application is then made, using integration, to a mixed harmonic—-analytic obstruction problem in §4. Finally, the Dirichlet deficiency of the analogue of the standard algebra on compact, nonorientable, Klein surfaces—with boundary—is computed. Again the defect of minus one occurs. Throughout, the reason why this defect occurs in the nonorientable case is of prime concern.

0. Foundations. The analytic foundations of the theory of Klein surfaces can be found in [11], [3], and [4]. Further, Greenleaf's paper [7], a companion to this, provides another reference, as well as proving Cartan's Theorem B in this context.

Let $\mathfrak{Y}$ be a noncompact, nonorientable, Klein surface, (without boundary), and let $\mathfrak{X} \xrightarrow{p} \mathfrak{Y}$ be its complex double [4]: i.e., $\mathfrak{X}$ is a Riemann surface and p a two-to-one local homeomorphism of $\mathfrak{X}$ onto $\mathfrak{Y}$. Recall also that $\mathfrak{X}$ has an antianalytic involution τ such that $p\circ\tau = p$. Let $\mathscr{O}$ be the structure sheaf on $\mathfrak{X}$, $\mathscr{C}$ the constant sheaf on $\mathfrak{X}$, and Ω the sheaf of germs of analytic differentials on $\mathfrak{X}$. Given an open set U of Y let $\mathscr{H}(U) \equiv \{s \in \Gamma(p^{-1}(U), \mathscr{O}): s = \sigma(s)\}$, where $\sigma(s) = \kappa\circ s\circ\tau$, κ denoting complex conjugation; thus $\mathscr{H}$ is a presheaf on Y. Let $\mathscr{K}$ be defined by replacing $\mathscr{O}$ by $\mathscr{C}$, $\mathscr{H}^*$ be replacing