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SPECTRAL THEORY OF MONOTONE HAMMERSTEIN OPERATORS

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Consider the linear integral equation,

$$y(t) = \mu \int_\Omega K(t, s) p(s) y(s)\, ds\,, \tag{1}$$

where $K(s, t)$ is a real-valued symmetric positive definite kernel and $p(s)$ is a positive function. Let L denote the inverse of the integral operator $u \longrightarrow \int_\Omega K(\cdot, s) u(s)\, ds$, and for a function y in the domain of L, $y \neq 0$, (all functions are assumed to be real valued) define the Rayleigh quotient $J(y)$ for (1) by,

$$J(y) = \int_\Omega y(t)\, [Ly](t)\, dt \Big/ \int_\Omega p(t) y^2(t)\, dt\,.$$

If $y_1 \neq 0$ and y_1 is in the domain of L and if

$$y_2 = \int_\Omega K(\cdot, s) p(s) y_1(s)\, ds\,,$$

then several applications of the Schwarz inequality show that,

$$J(y_2) \leqq J(y_1)\,,$$

with equality only if y_1 is an eigenfunction of (1). On the basis of this fact, when the integral operator in (1) is compact, one can develop the complete spectral theory of (1).

In this paper it is shown that the approach indicated above for the study of (1) has a simple and natural extension for the study of the nonlinear integral equation,

$$y(t) = \mu \int_\Omega K(t, s) f(s, y(s))\, ds\,, \tag{2}$$

where $K(t, s)$ is as above and $f(t, y)$ is an odd function of y,

$$f(t, y) = -f(t, -y)\,,$$

and satisfies,

$$yf(t, y) > 0\,, \qquad y \neq 0\,,$$

and

$$f(t, y_2) \geqq f(t, y_1)\,, \qquad y_2 \geqq y_1\,.$$

The problem of minimizing the Rayleigh quotient $J(y)$ for (1) can be generalized to either of the dual variational problems,