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OPERATORS THAT COMMUTE WITH A UNILATERAL SHIFT ON AN INVARIANT SUBSPACE

LAVON B. PAGE

A co-isometry on a Hilbert space $\mathscr{H}$ is a bounded operator having an isometric adjoint. If V is a co-isometry on $\mathscr{H}$ and $\mathscr{M}$ is an invariant subspace for V, then every bounded operator on $\mathscr{M}$ that commutes with V on $\mathscr{M}$ can be extended to an operator on $\mathscr{H}$ that commutes with V, and the extension can be made without increasing the norm of the operator. This paper is concerned with unilateral shifts. The questions asked are these: (1) Do shifts enjoy the above property shared by co-isometries and self-adjoint operators? (The answer to this question is "rarely".) (2) Why not? (3) If S is a shift, $\mathscr{M}$ is an invariant subspace for S, S_0 is the restriction of S to $\mathscr{M}$, and T is a bounded operator on $\mathscr{M}$ satisfying $TS_0 = S_0T$, how tame do T and $\mathscr{M}$ have to be in order that T can be extended (without increasing the norm) to an operator in the commutant of S? Extension is possible in a large number of cases.

The result mentioned above for co-isometries is due to Sz.-Nagy and Foias [8]. (An excellent exposition on the problem is found in [3]; see Theorem 4 in particular.) For self-adjoint operators the statement is trivial for the simple reason that every invariant subspace is then reducing and any commuting operator on a subspace can be extended by simply requiring it to be zero on the orthogonal complement of the subspace.

Recall that a unilateral shift S is an isometry having the property that $\bigcap_{n=0}^{\infty} S^n\mathscr{H} = \{0\}$. The Hilbert space dimension of the subspace $(S\mathscr{H})^{\perp}$ is called the multiplicity of S. Within the class of partial isometries on $\mathscr{H}$ the unilateral shifts are in a sense as far removed as possible from the co-isometries and the self-adjoint partial isometries. For shifts have no self-adjoint part, and far from being co-isometric if S is a shift S^{*n} goes strongly to zero. (These and other simple properties of shifts may be deduced from problem 118 and the surrounding material in Halmos [5].)

II. We begin with a complex Hilbert space $\mathscr{H}$ (not necessarily separable) and a unilateral shift S on $\mathscr{H}$. It is well known that shifts decompose the underlying Hilbert space in the following way:

$$\mathscr{H} = \bigoplus\sum_{n=0}^{\infty} S^n\mathscr{C} \quad \text{where} \quad \mathscr{C} = (S\mathscr{H})^{\perp} ,$$