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QUASI-PROJECTIVE AND QUASI-INJECTIVE MODULES

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This paper contains results which are needed to prove a decomposition theorem for quasi-projective modules over left perfect rings.

An R-module M is called quasi-projective if and only if for every R-module A, every R-epimorphism $q: M \to A$, and every R-homomorphism $f: M \to A$, there is an $f' \in \operatorname{End}_R(M)$ such that the diagram

$$\begin{array}{ccccc} & & M & & \\ & {\scriptstyle f'}\swarrow & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{q} & A & \longrightarrow & 0 \end{array}$$

commutes, that is, $q \circ f' = f$. An R-module M is called quasi-injective if and only if for every R-module A, every R-monomorphism $j: A \to M$, and R-homomorphism $f: A \to M$, there is an $f' \in \operatorname{End}_R(M)$ such that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{j} & M \\ & & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle f'} & \\ & & M & & \end{array}$$

commutes.

The first section of this paper contains results which are needed to prove a decomposition theorem for quasi-projective modules over left perfect rings (Theorem 1.10). This decomposition is a characterization for quasi-projective modules over left perfect rings. A ring is left perfect if a projective cover (the dual concept of injective envelope) exists for every left R-module [4, p. 467]. It is known, for example, that left Artinian rings are left perfect [4, p. 467]. Some of the propositions are stated for semiperfect rings which are rings such that every finitely generated module has a projective cover [4, p. 471].

In the second section the decomposition for quasi-projective modules is used to obtain a decomposition for quasi-injective modules over a special class of rings. For these rings this decomposition characterizes quasi-injective modules. This decomposition theorem (Theorem 2.5) is specialized to the cases where the ring is quasi-Frobenius and where it is a finite dimensional algebra.

It will be assumed that all rings have an identity and that the