PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

B-SETS AND PLANAR MAPS

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In this paper we examine the relation between B-sets, which are a purely set-theoretic concept, and various concepts associated with planar maps, for instance, four-colorings, five-colorings, Hamiltonian circuits, and Petersen's theorem. Moreover, the introduction of the notion of a B-set into graph theory enables us to ask questions which may be more tractable than the four-color conjecture and shed light on it.

1. Introduction and definitions. Let F be a family of sets. A set that meets every member of F and yet contains none of the members of F is called a B-set for F. Observe that if B is a B-set for F, then so is its complement, $(\bigcup_{E \in F} E) - B$. In fact, F has a E-set if and only if $\bigcup_{E \in F} E$ can be partitioned into two sets, A and B, such that neither A nor B contains a member of F. Observe that if F has a B-set, and if $G \subseteqq F$, then G has a B-set. Also, if $G \subseteqq F$, and every member of F contains some member of G, and if G has a B-set, then F also has a B-set. (The notion of B-set goes back to Bernstein, who used it in 1908 to deal with a topological question.)

We shall be concerned with maps covering the surface of the sphere, S^2. For the most part, we will assume that these maps are 3-regular, that is, each vertex has degree three. Each region of the map will be a topological cell. Two regions are *adjacent* if they share at least one edge.

A sequence of distinct regions $R_1, R_2, \cdots, R_{n+1}$, $n \geqq 1$ such that R_i is adjacent to R_{i+1}, $1 \leqq i \leqq n$ is a *path of regions*. If we have $R_{n+1} = R_1$, and $R_1, R_2, \cdots R_n$ are still distinct, we call the path a *region-cycle* of length n (or *n-region-cycle*). A region-cycle consisting simply of the regions around a vertex we call a *basic cycle*. In a 3-regular map the basic cycles have length three.

If the union of any two regions in a map is simply connected, then the regions bordering any given region form a region-cycle, which we call a *face cycle*. Its length is just the number of edges of the surrounded region.

We shall not be interested in region-cycles of length two, unless their union is not simply connected. A region-cycle is *odd* if its length is odd; otherwise, it is *even*.

2. B-sets, four-coloring, and Hamiltonian Circuits. It is well known that the vertices of a graph can be colored in four colors if and only if the set of vertices can be partitioned into two sets such