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AN INEQUALITY FOR THE HILBERT TRANSFORM

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The purpose of this paper is to give a general inequality for the Hilbert transform that clearly distinguishes conditions of global and local integrability. The former conditions are associated with a certain product $[f, g]_m$, the latter with another product $\{f, g\}_p$. The resulting statement contains, as corollaries, a number of inequalities for the Hilbert transform that have not been hitherto noted. Presentation of these is a second objective here. It turns out that the general theorem also includes, in sharpened form, several classical inequalities of Hardy and Littlewood, Babenko, and others. Proof of these sharpened forms is a third objective.

By means of the theory of Calderón and Zygmund results similar to those of this paper can be established for Hilbert transforms in n dimensions and for singular integrals of more general types. However, this is not done here.

1. Definitions and notation. We use f, g, h and so on for complex-valued functions of a real variable, u or x, on the domain

$$(-\infty, 0) \cup (0, \infty)\,.$$

All functions are assumed locally integrable, that is, integrable in the sense of Lebesgue over each compact subinterval of the above domain, and p and q are complementary Lebesgue exponents. Thus,

$$p \geqq 1, \quad \frac{1}{p} + \frac{1}{q} = 1\,.$$

The statement $f \in L^p$ means that f is locally integrable and $|f|^p$ is integrable on $(-\infty, \infty)$, the omitted point 0 being irrelevant here. Otherwise, our integrals are interpreted whenever possible as Cauchy principal values near $0, x$ and $\pm\infty$. This applies, in particular, to the Hilbert transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} \frac{f(u)}{x - u}\, du$$

and to the modified Hilbert transform $\hat{f}_m$ introduced below.

We define, as usual, $L = L^1$, and

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx\right)^{1/p}, \qquad p \geqq 1\,.$$

Any inequality of form $P \leqq Q$ is considered to be vacuously fulfilled