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STRICTLY INCREASING RIESZ NORMS

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Let L be a Riesz space and ρ a Riesz norm on L. Then ρ is said to be *strictly increasing* if $u, v \in L$ and $0 \leqq u \lneqq v$ imply that $\rho(u) < \rho(v)$. We investigate necessary conditions and sufficient conditions that for a given Riesz norm there is an equivalent strictly increasing Riesz norm. A necessary condition is that the Riesz space possess the countable sup property. A sufficient condition is that the given norm be an (A, ii) norm. Finally, we investigate the relationship between the existence of strictly increasing Riesz norms and the Souslin hypothesis.

For reference, we list here several definitions. If ρ is a Riesz norm on a Riesz space L, then ρ^* is the dual norm on the Riesz space of ρ-bounded linear functionals. A Riesz norm ρ is said to have property (A, ii) if whenever $\{u_\lambda\}$ is directed downwards in L and $\inf_\lambda u_\lambda = 0$, then $\inf_\lambda \rho(u_\lambda) = 0$. An order bounded linear functional φ on L is said to be an *integral* if $\{u_n\}$ is a sequence in L and $u_n \downarrow 0$ implies $\inf_n |\varphi(u_n)| = 0$. If whenever $\{u_\lambda\}$ is a set in L which is directed downwards and $\inf_\lambda u_\lambda = 0$, then $\inf_\lambda |\varphi(u_\lambda)| = 0$, we say that φ is a *normal integral*. If ρ is a Riesz norm on L, then $L^*_{\rho,c}$ is the Riesz space of all ρ-bounded integrals and $L^*_{\rho,n}$ is the Riesz space of all ρ-bounded normal integrals. A Riesz space L is said to have the *countable sup property* if every nonempty subset D of L possessing a supremum contains an at most countable subset having the same supremum as D.

Luxemburg discussed the countable sup property in [3] and, in the case that the Riesz space is Archimedean, gave a number of equivalent conditions (Theorem 6 of [3]). In addition he showed that if L is Dedekind σ-complete and possesses a strictly increasing Riesz norm, then L has the countable sup property (Theorem 10 of [3]). In a private communication, Luxemburg asked whether the theorem is true if the condition that L be Dedekind σ-complete is dropped. The answer is yes and is the content of Corollary 2 of this paper. That an (A, ii) norm has an equivalent strictly increasing Riesz norm (Theorem 5) was suggested by Luxemburg, and, in the case that L is countably generated, it appears in [4].

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2. Necessary conditions and sufficient conditions.

If a Riesz space L with Riesz norm ρ possesses a ρ-bounded strictly positive